Computing the Mixed Concept Lattice

Francisco Pérez-Gámez[1(✉)], Pablo Cordero[1], Manuel Enciso[2], Domingo López-Rodríguez[1], and Ángel Mora[1]

[1] Departamento de Matemática Aplicada, Universidad de Málaga, Málaga, Spain
{franciscoperezgamez,pcordero,dominlopez,amora}@uma.es

[2] Departamento de Lenguajes y Ciencias de la Computación, Universidad de Málaga, Málaga, Spain
enciso@uma.es

Abstract. The classical approach on Formal Concept Analysis (FCA) extracts knowledge from a binary table $\mathbb{K} = (G, M, I)$ taking into account the existing relationships (given by the binary relation I) between objects G and attributes M. Thus, this classical setting accounts only for *positive information*. Particularly, FCA allows to define and compute the concept lattice $\underline{\mathbb{B}}(\mathbb{K})$ from this positive information. As an extension of this framework, some works consider not only this positive information, but also the negative information that is explicit when objects have no relation to specific attributes (denoted by $\overline{\mathbb{K}}$). These works, therefore, use the apposition of *positive* and *negative* information to compute the *mixed* concept lattice $\underline{\mathbb{B}}^{\#}(\mathbb{K})$. In this paper, we propose to establish the relationships between extents and intents of concepts in $\underline{\mathbb{B}}(\mathbb{K})$, $\underline{\mathbb{B}}(\overline{\mathbb{K}})$ and $\underline{\mathbb{B}}^{\#}(\mathbb{K})$ and how to address an incremental algorithm to compute $\underline{\mathbb{B}}^{\#}(\mathbb{K})$ merging the knowledge on $\underline{\mathbb{B}}(\mathbb{K})$, $\underline{\mathbb{B}}(\overline{\mathbb{K}})$ previously obtained with classical methods.

Keywords: Formal concept analysis · Mixed attributes · Concept lattice

1 Introduction

In the classical paradigm of formal concept analysis (FCA) [4,5], the fundamental data model is a structure (called formal context) that represents a binary relationship between a set of objects and their attributes. From this formal context, we can define formal concepts, which represent sets of objects that share common attributes. In addition, we can define an ordering relationship between concepts in the sense of *specialisation-generalisation*, which endows such a set of

Partially supported by the Spanish Ministry of Science, Innovation, and Universities (MCIU), State Agency of Research (AEI), Junta de Andalucía (JA), Universidad de Málaga (UMA) and European Regional Development Fund (FEDER) through the projects PGC2018-095869-B-I00 (MCIU/AEI/FEDER), TIN2017-89023-P (MCIU/AEI/FEDER), PRE2018-085199 and UMA2018-FEDERJA-001 (JA/UMA/FEDER).

D. Ciucci et al. (Eds.): IPMU 2022, CCIS 1601, pp. 87–99, 2022.
https://doi.org/10.1007/978-3-031-08971-8_8

concepts with the structure of a complete lattice. This paradigm has been used successfully in problems such as the construction of recommender systems [1,3] or in the analysis of data from social networks [2,7,9].

According to the classical scheme, FCA is not suitable for handling negative information (indicating the absence of some property in an object, in contrast to the presence which is assumed by default to be positive information). But some practical applications require such information to be explicitly contemplated, hence the need to study from a formal perspective the mixture of the two types of information.

Early attempts to handle mixed information in FCA tried to model the problem by apposing the *positive* context and its negation [6,10], so that the number of attributes is doubled and, therefore, the computational treatment of the problem has a higher complexity. Moreover, as mentioned in [11], real applications often have sparse data in the positive context and are very dense in the negative context, or vice versa, which generates a large amount of redundancy in the expression of information. Some works deal with positive and negative information [8,11,13,14] more efficiently, but they do not study the relationship between the different types of information at the concept lattice level.

In the present proposal, we build on the line of work of [13,14], in which, instead of relying on the apposition of positive and negative contexts, the concept formation operators, i.e. the Galois connection inducing a closure system, are redefined to allow for the mixing of information types without the need to duplicate the context. The treatment of mixed information under a unified framework improves the expressiveness of the model significantly.

The main aim of the paper is to relate the closed sets of the positive and negative contexts to those that can be obtained by means of the new derivation operators, which we denote by $\Uparrow$ and $\Downarrow$, instead of the traditional $\uparrow$ and $\downarrow$. To do so, we will define *embedding* and *projection* operators, which represent, in turn, a Galois connection between the individual lattices and the mixed lattice.

Thanks to these operators, we can demonstrate the isomorphism between the individual lattices and sub-semilattices of the mixed lattice, and the relationship between the extents and intents of the different lattice types. In addition, we will present results that establish the decomposition or representation of a mixed concept in terms of a pair of suitable positive and negative concepts. Thanks to these theoretical results, we will be able to propose an algorithm for the computation of the lattice of mixed concepts from the positive and negative lattices.

The rest of this work is structured as follows: in Sect. 2, we find the preliminary notions and definitions that will be used throughout this work. In Sect. 3, we present the main theoretical results that relate the individual concept lattices to the mixed lattice, and we provide an incremental algorithm able to compute this mixed lattice from the individual concept lattices. Finally, in Sect. 4, we present the conclusions and the proposal of future works in this line.

2 Preliminary Notions

Formal concept Analysis (FCA) is a useful tool to extract knowledge from a collection of data stored on a formal context $\mathbb{K} = (G, M, I)$, where G is a set of objects, M is a set of attributes and I represents the relation between them.

In a formal context $\mathbb{K} = (G, M, I)$, we can define a Galois connection used to extract the concepts behind the data. The Galois connection is formed by the mappings ${}^{\uparrow} : 2^G \to 2^M$ and ${}^{\downarrow} : 2^M \to 2^G$ defined in the following way. $A^{\uparrow} = \{m \in M \mid (g, m) \in I \text{ for all } g \in A\}$, that is, all the attributes shared by the objects in A and $B^{\downarrow} = \{g \in G \mid (g, m) \in I \text{ for all } m \in B\}$, i.e., all the objects that satisfy all the attributes in B. In [15] it is proved that these two maps form a Galois connection, therefore their compositions ${}^{\uparrow\downarrow}$ and ${}^{\downarrow\uparrow}$ are closure operators.

Using these notions we are able to extract knowledge: We say that a pair (A, B) with $A \subseteq G$ and $B \subseteq M$ is a formal concept if $A^{\uparrow} = B$ and $B^{\downarrow} = A$. So (A, B) is a formal context if all the objects in A share all the attributes in B and they do not share any other attributes. In addition, given two formal concepts (A, B) and (C, D) we can define a order relation between then with $(A, B) \leq (C, D)$ if and only if $A \subseteq C$ or, equivalently, if and only if $D \subseteq B$. It is well-known that the set with all formal concepts together with the order form a complete lattice that will be denoted by $\underline{\mathbb{B}}(\mathbb{K})$.

The classical FCA paradigm only consider the information provided by the incidence relation I, and does not model the *negative* information contained therein, that is, the information provided by the pairs $(g, m) \notin I$. In this paper we want to extend to consider the positive and the negative information. Recently some authors work with this view like [8,11,13,14]. In [11] we find an approach that, given a formal context $\mathbb{K} = (G, M, I)$, they built a new one $(\mathbb{K} \mid \overline{\mathbb{K}}) = (G, M \cup \overline{M}, I^*)$ being $I^*(g, m) = I(g, m)$ for all $m \in M$ and $I^*(g, \overline{m}) = \min(1, 1 - I(g, m))$ otherwise. Here, the attributes in M are called positive attributes and the attributes in $\overline{M}$ are called negative attributes. This approach duplicates the number of attributes so the methods loses efficiency.

Here, we follow the working line of [14] which, instead of duplicating the number of attributes, define a new Galois connection over $\mathbb{K}$, i.e., the formal context does not change. The new connection is denoted by ${}^{\Uparrow}$ and ${}^{\Downarrow}$ to difference them from the Galois connection defined before. We define mixed context as any formal context $\mathbb{K}$ provided with the new Galois connection. We define the new operators ${}^{\Uparrow} : 2^G \to 2^{M \cup \overline{M}}$ and ${}^{\Downarrow} : 2^{M \cup \overline{M}} \to 2^G$ as follows:

$$X^{\Uparrow} = \{m \in M \mid (g, m) \in I \ \ \forall\, g \in X\} \cup \{\overline{m} \in \overline{M} \mid (g, m) \notin I \ \ \forall\, g \in X\}$$

$$Y^{\Downarrow} = \{g \in G \mid (g, m) \in I \ \ \forall\, m \in Y\} \cap \{g \in G \mid (g, m) \notin I \ \ \forall\, \overline{m} \in Y\}$$

These two operators form a Galois connection and, as consequence, both compositions are closure operators. Consequently, using the closure operators we can build a concept lattice over the mixed context of $\mathbb{K}$. We denote by $\underline{\mathbb{B}}^{\#}(\mathbb{K})$ the lattice built by using the derivation operators ${}^{\Uparrow}$ and ${}^{\Downarrow}$, and we denote by $\underline{\mathbb{B}}(\mathbb{K})$ the concept lattice formed by using the classic operators ${}^{\uparrow}$ and ${}^{\downarrow}$. Observe that with this new derivation operators we obtain the same concept lattice that the concept lattice build with the formal context with the double of attributes so we can process the same information without duplicating the number of columns and, as consequence, this view is more efficient.

Let A be a set of attributes, now we present some functions that allow us to capture the positive and negative information related to this set:

$$\mathrm{Pos}(A) = A \cap M, \qquad \mathrm{Neg}(A) = \overline{A} \cap M$$

where $\overline{A} = \{\overline{m} \in \overline{M} : m \in A\} \cup \{m \in M : \overline{m} \in A\}$.

Given a formal context $\mathbb{K}$ we define its complement as $\overline{\mathbb{K}} = (G, M, \overline{I})$ being $\overline{I}$ defined by $(g, m) \in \overline{I}$ if and only if $(g, m) \notin I$.

Lemma 1 **([12]).** *Let* $\mathbb{K}$ *be a formal context. The following statements are fulfilled:*

1. *If* $A \subseteq M$*, then* $A^{\Downarrow} = A^{\downarrow}$ *(in* $\mathbb{K}$*).*
2. *If* $A \subseteq \overline{M}$*, then* $A^{\Downarrow} = A^{\downarrow}$ *(in* $\overline{\mathbb{K}}$*).*
3. *If* $B \subseteq G$*, then* $\mathrm{Pos}(B^{\Uparrow}) = B^{\uparrow}$ *(in* $\mathbb{K}$*) and* $\mathrm{Neg}(B^{\Uparrow}) = B^{\uparrow}$ *(in* $\overline{\mathbb{K}}$*).*

The following example will be used to show the main results in this work.

Example 1. Let $\mathbb{K}$ be a formal context given by Table 1(a). The apposition (concatenation by columns) of $\mathbb{K}$ and its complement $\overline{\mathbb{K}}$ is in Table 1(b). As usual, we follow the notation $\mathbb{K}|\overline{\mathbb{K}}$ for the concatenated formal context. In addition, we define $M = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$ and $\overline{M} = \{\overline{\mathrm{a}}, \overline{\mathrm{b}}, \overline{\mathrm{c}}, \overline{\mathrm{d}}\}$.

Table 1. Formal contexts for the example.

(a)

	a	b	c	d
o1		×		×
o2	×	×		
o3	×			×
o4			×	
o5		×	×	×
o6	×		×	
o7		×	×	

(b)

	a	b	c	d	$\overline{\mathrm{a}}$	$\overline{\mathrm{b}}$	$\overline{\mathrm{c}}$	$\overline{\mathrm{d}}$
o1		×		×	×		×	
o2	×	×					×	×
o3	×			×		×	×	
o4			×		×	×		×
o5		×	×	×	×			
o6	×		×			×		×
o7		×	×		×			×

The following example shows the difference between the two Galois connection defined over $\mathbb{K}$, that is, between $({}^{\uparrow}, {}^{\downarrow})$ and $({}^{\Uparrow}, {}^{\Downarrow})$.

$$\{\mathrm{c}, \mathrm{d}\}^{\downarrow\uparrow} = \left(\{\mathrm{c}, \mathrm{d}\}^{\downarrow}\right)^{\uparrow} = \{\mathrm{o5}\}^{\uparrow} = \{\mathrm{b}, \mathrm{c}, \mathrm{d}\}$$

$$\{\mathrm{c}, \mathrm{d}\}^{\Downarrow\Uparrow} = \left(\{\mathrm{c}, \mathrm{d}\}^{\Downarrow}\right)^{\Uparrow} = \{\mathrm{o5}\}^{\Uparrow} = \{\mathrm{b}, \mathrm{c}, \mathrm{d}, \overline{\mathrm{a}}\}$$

Furthermore, we can check the results of Lemma 1, since $\{\mathrm{c}, \mathrm{d}\}^{\Downarrow} = \{\mathrm{c}, \mathrm{d}\}^{\downarrow} = \{\mathrm{o5}\}$ and $\mathrm{Pos}(\{\mathrm{o5}\}^{\Uparrow}) = \mathrm{Pos}(\{\mathrm{b}, \mathrm{c}, \mathrm{d}, \overline{\mathrm{a}}\}) = \{\mathrm{b}, \mathrm{c}, \mathrm{d}\} = \{\mathrm{o5}\}^{\uparrow}$.

Finally, we can see that ${}^{\downarrow}$ is defined on 2^{M} whereas ${}^{\Downarrow}$ is in $2^{M \cup \overline{M}}$ so it does not make sense to write $\{\mathrm{c}, \overline{\mathrm{b}}\}^{\downarrow}$. Instead, we have to rely on ${}^{\Downarrow}$ to compute the desired *extent* using mixed attributes: $\{\mathrm{c}, \overline{\mathrm{b}}\}^{\Downarrow} = \{\mathrm{o4}, \mathrm{o6}\}$.

3 Main Results

Hereinafter, we will consider a formal context $\mathbb{K} = (G, M, I)$ and its *negative* context $\overline{\mathbb{K}}$, both equipped with their corresponding concept-forming operators $(^{\downarrow}, ^{\uparrow})$, and the mixed context equipped with $(^{\Downarrow}, ^{\Uparrow})$. The main objective of this work is to characterise the elements of the mixed lattice $\underline{\mathbb{B}}^{\#}(\mathbb{K})$ in terms of those of the positive and negative lattices.

We divide this study into two strategies. The first part of this section will be devoted to studying how the individual positive and negative lattices can be mapped into the mixed lattice, analysing their inclusion and the isomorphism relationship between these lattices and sub-semilattices of the $\underline{\mathbb{B}}^{\#}(\mathbb{K})$. The second part will focus on analysing how to decompose the concepts of the mixed lattice in terms of concrete concepts of the individual lattices, allowing the definition of the algorithm for the computation of the mixed lattice from the latter.

3.1 Embedding of $\underline{\mathbb{B}}(\mathbb{K})$ and $\underline{\mathbb{B}}(\overline{\mathbb{K}})$ into $\underline{\mathbb{B}}^{\#}(\mathbb{K})$

Here, as mentioned above, we present theoretical results that allow us to study the inclusion relation of the positive and negative lattices within the $\underline{\mathbb{B}}^{\#}(\mathbb{K})$. We will start by studying how the extents of the mixed lattice have a higher level of granularity than the extents of the individual lattices. We will use the following notation in order to make the results of this work easier to read:

Notation 1. *Let us denote:*

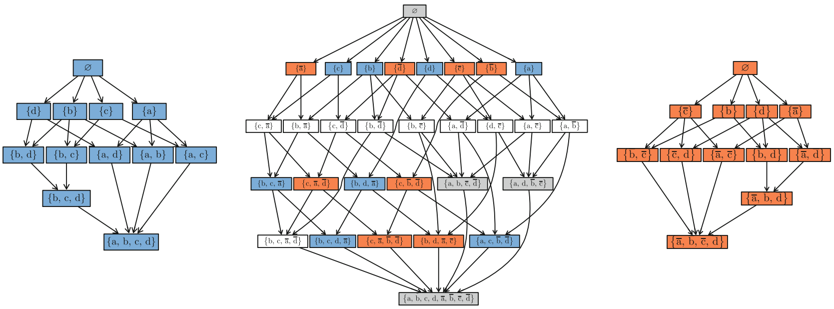
$\mathrm{Ext}(\mathbb{K}) := \{A \subseteq G : A \textit{ is the extent of a concept in } \underline{\mathbb{B}}(\mathbb{K})\}$
$\mathrm{Int}(\mathbb{K}) := \{B \subseteq M : A \textit{ is the intent of a concept in } \underline{\mathbb{B}}(\mathbb{K})\}$
$\mathrm{Ext}^{\#}(\mathbb{K}) := \{A \subseteq G : A \textit{ is the extent of a concept in } \underline{\mathbb{B}}^{\#}(\mathbb{K})\}$
$\mathrm{Int}^{\#}(\mathbb{K}) := \{B \subseteq M \cup \overline{M} : A \textit{ is the intent of a concept in } \underline{\mathbb{B}}^{\#}(\mathbb{K})\}$

With this notation, we refer to the set of extents and intents of concepts in $\underline{\mathbb{B}}(\mathbb{K})$ and $\underline{\mathbb{B}}^{\#}(\mathbb{K})$, respectively. Analogously, we can denote the corresponding ones in the negative lattice $\overline{\mathbb{K}}$.

Lemma 2. *For a given formal context* $\mathbb{K} = (G, M, I)$*, we have* $\mathrm{Ext}(\mathbb{K}) \subseteq \mathrm{Ext}^{\#}(\mathbb{K})$ *and* $\mathrm{Ext}(\overline{\mathbb{K}}) \subseteq \mathrm{Ext}^{\#}(\mathbb{K})$.

We continue the example above to give a graphical representation of the situation stated in the previous theoretical result.

Example 2. We continue with the same contexts of Example 1. In Fig. 1, we show the concept lattices $\underline{\mathbb{B}}(\mathbb{K})$, $\underline{\mathbb{B}}(\overline{\mathbb{K}})$ and $\underline{\mathbb{B}}^{\#}(\mathbb{K})$ of the *positive*, *negative* and mixed formal contexts. We have used a colour code to represent the relationships of $\mathrm{Ext}(\mathbb{K})$ and $\mathrm{Ext}(\overline{\mathbb{K}})$ with $\mathrm{Ext}^{\#}(\mathbb{K})$. We have marked in blue the concepts whose extents are present in $\underline{\mathbb{B}}(\mathbb{K})$, and in orange those present in $\underline{\mathbb{B}}(\overline{\mathbb{K}})$. Note that some concepts are in grey since their extent appears in both $\underline{\mathbb{B}}(\mathbb{K})$ and $\underline{\mathbb{B}}(\overline{\mathbb{K}})$. We can observe that the concepts in the mixed context contain those of the other two contexts and provide additional information and granularity.



(a) (b) (c)

Fig. 1. Concept lattices associated with (a) the *positive* context $\mathbb{K}$ in Table 1(a); (b) the mixed context; (c) the *negative* formal context $\overline{\mathbb{K}}$.

The Lemma 2 allows us to define the following operators:

$$\begin{array}{ll|ll} e_+ : \underline{\mathbb{B}}(\mathbb{K}) & \to \underline{\mathbb{B}}^{\#}(\mathbb{K}) & e_- : \underline{\mathbb{B}}(\overline{\mathbb{K}}) & \to \underline{\mathbb{B}}^{\#}(\mathbb{K}) \\ (A, B) & \mapsto e_+(A, B) := (A, A^{\Uparrow}) & (A, B) & \mapsto e_-(A, B) := (A, A^{\Uparrow}) \end{array}$$

Note that these mappings are well-defined since, by Lemma 2, every extent of $\underline{\mathbb{B}}(\mathbb{K})$ and $\underline{\mathbb{B}}(\overline{\mathbb{K}})$ is also an extent of $\underline{\mathbb{B}}^{\#}(\mathbb{K})$. We can say that these operators define *embeddings* of $\underline{\mathbb{B}}(\mathbb{K})$ and of $\underline{\mathbb{B}}(\overline{\mathbb{K}})$ into $\underline{\mathbb{B}}^{\#}(\mathbb{K})$. We shall now study their properties:

Theorem 1. *For a formal context $\mathbb{K} = (G, M, I)$, the embedding mappings e_+ and e_- are $\wedge$-preserving and injective.*

Let us now recall the projection operators $\pi_+ : \underline{\mathbb{B}}^{\#}(\mathbb{K}) \to \underline{\mathbb{B}}(\mathbb{K})$ and $\pi_- : \underline{\mathbb{B}}^{\#}(\mathbb{K}) \to \underline{\mathbb{B}}(\overline{\mathbb{K}})$ introduced in [12] and defined as

$$\pi_+(A, B) := (\mathrm{Pos}(B)^{\Downarrow}, \mathrm{Pos}(B)) \quad \Big| \quad \pi_-(A, B) := (\overline{\mathrm{Neg}(B)}^{\Downarrow}, \overline{\mathrm{Neg}(B)})$$

For these mappings, we have the following result:

Theorem 2 ([12]). *The maps $\pi_+ : \underline{\mathbb{B}}^{\#}(\mathbb{K}) \to \underline{\mathbb{B}}(\mathbb{K})$ and $\pi_- : \underline{\mathbb{B}}^{\#}(\mathbb{K}) \to \underline{\mathbb{B}}(\overline{\mathbb{K}})$ are $\vee$-preserving and surjective.*

We can go even further in the study of their properties. According to the following theoretical result, these *embedding* and *projection* operators establish Galois connections between the individual lattices ($\underline{\mathbb{B}}(\mathbb{K})$ and $\underline{\mathbb{B}}(\overline{\mathbb{K}})$) and the mixed lattice.

Theorem 3. *For a formal context $\mathbb{K} = (G, M, I)$, we have:*

1. e_+, e_-, π_+ and π_- are monotone.
2. $\pi_+ \circ e_+ = \mathrm{id}_{\underline{\mathbb{B}}(\mathbb{K})}$ and $\pi_- \circ e_- = \mathrm{id}_{\underline{\mathbb{B}}(\overline{\mathbb{K}})}$.
3. (e_+, π_+) and (e_-, π_-) are isotone Galois connections.

In our case, (e_+, π_+) and (e_-, π_-) are not only Galois connections: by Theorem 3, $\pi_+ \circ e_+$ and $\pi_- \circ e_-$ are the identity mappings in $\underline{\mathbb{B}}(\mathbb{K})$ and $\underline{\mathbb{B}}(\overline{\mathbb{K}})$, respectively, and this condition allows us to say that they are **Galois injections** (also called Galois surjections). Let us introduce the following notation:

Notation 2. *Consider a formal context $\mathbb{K} = (G, M, I)$ and the operators e_+, e_-, π_+ and π_- defined as above. Then, we denote $\sigma_+ := e_+ \circ \pi_+$ and $\sigma_- := e_- \circ \pi_-$, where both maps are $\underline{\mathbb{B}}^{\#}(\mathbb{K}) \to \underline{\mathbb{B}}^{\#}(\mathbb{K})$.*

As a consequence of Theorem 3, we have:

Corollary 1. *σ_+ and σ_- are closure operators in $\underline{\mathbb{B}}^{\#}(\mathbb{K})$.*

We now finish the study of the embeddings of the individual lattices within the mixed lattice by showing that the system of closures induced by the above closure operators coincides with the embeddings of the individual lattices within $\underline{\mathbb{B}}^{\#}(\mathbb{K})$. Let us denote $\mathcal{C}_+$ and $\mathcal{C}_-$ the set of closed sets according to operators σ_+ and σ_-, respectively. Then, we have:

Proposition 1. *The closed sets of σ_+ and σ_- are the embeddings of $\underline{\mathbb{B}}(\mathbb{K})$ and $\underline{\mathbb{B}}(\overline{\mathbb{K}})$ into $\underline{\mathbb{B}}^{\#}(\mathbb{K})$, respectively. That is*

$$\mathcal{C}_+ = e_+(\underline{\mathbb{B}}(\mathbb{K})) \quad \textit{and} \quad \mathcal{C}_- = e_-(\underline{\mathbb{B}}(\overline{\mathbb{K}}))$$

Example 3. We continue with the formal context presented in Example 1. Here, we present in Table 2 the embeddings of the lattices of $\mathbb{K}$ and $\overline{\mathbb{K}}$. In the table, we only list the intents of the corresponding concepts for the sake of clarity.

Table 2. Embedding of the individual concept lattices $\underline{\mathbb{B}}(\mathbb{K})$ and $\underline{\mathbb{B}}(\overline{\mathbb{K}})$ into $\underline{\mathbb{B}}^{\#}(\mathbb{K})$. Only intents are shown in this table.

$(A, B) \in \underline{\mathbb{B}}(\mathbb{K})$	$e_+(A, B)$	$(A, B) \in \underline{\mathbb{B}}(\overline{\mathbb{K}})$	$e_-(A, B)$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
{d}	{d}	$\{\overline{\text{d}}\}$	$\{\overline{\text{d}}\}$
{c}	{c}	$\{\overline{\text{c}}\}$	$\{\overline{\text{c}}\}$
{b}	{b}	$\{\overline{\text{c}}, \overline{\text{d}}\}$	$\{\text{a}, \text{b}, \overline{\text{c}}, \overline{\text{d}}\}$
{b, d}	$\{\text{b}, \text{d}, \overline{\text{a}}\}$	$\{\overline{\text{b}}\}$	$\{\overline{\text{b}}\}$
{b, c}	$\{\text{b}, \text{c}, \overline{\text{a}}\}$	$\{\overline{\text{b}}, \overline{\text{d}}\}$	$\{\text{c}, \overline{\text{b}}, \overline{\text{d}}\}$
{b, c, d}	$\{\text{b}, \text{c}, \text{d}, \overline{\text{a}}\}$	$\{\overline{\text{b}}, \overline{\text{c}}\}$	$\{\text{a}, \text{d}, \overline{\text{b}}, \overline{\text{c}}\}$
{a}	{a}	$\{\overline{\text{a}}\}$	$\{\overline{\text{a}}\}$
{a, d}	$\{\text{a}, \text{d}, \overline{\text{b}}, \overline{\text{c}}\}$	$\{\overline{\text{a}}, \overline{\text{d}}\}$	$\{\text{c}, \overline{\text{a}}, \overline{\text{d}}\}$
{a, c}	$\{\text{a}, \text{c}, \overline{\text{b}}, \overline{\text{d}}\}$	$\{\overline{\text{a}}, \overline{\text{c}}\}$	$\{\text{b}, \text{d}, \overline{\text{a}}, \overline{\text{c}}\}$
{a, b}	$\{\text{a}, \text{b}, \overline{\text{c}}, \overline{\text{d}}\}$	$\{\overline{\text{a}}, \overline{\text{b}}, \overline{\text{d}}\}$	$\{\text{c}, \overline{\text{a}}, \overline{\text{b}}, \overline{\text{d}}\}$
{a, b, c, d}	$\{\text{a}, \text{b}, \text{c}, \text{d}, \overline{\text{a}}, \overline{\text{b}}, \overline{\text{c}}, \overline{\text{d}}\}$	$\{\overline{\text{a}}, \overline{\text{b}}, \overline{\text{c}}, \overline{\text{d}}\}$	$\{\text{a}, \text{b}, \text{c}, \text{d}, \overline{\text{a}}, \overline{\text{b}}, \overline{\text{c}}, \overline{\text{d}}\}$

Finally, we state a result that is direct consequence of Corollary 1 and Proposition 1.

Corollary 2. *$\underline{\mathbb{B}}(\mathbb{K})$ and $\underline{\mathbb{B}}(\overline{\mathbb{K}})$ are isomorphic to $\wedge$-subsemilattices of $\underline{\mathbb{B}}^{\#}(\mathbb{K})$.*

3.2 Decomposition of Concepts in $\underline{\mathbb{B}}^{\#}(\mathbb{K})$

In this part, we approach the problem of the decomposition or representation of mixed concepts in terms of concepts from the positive and negative lattices. The first result uses the closure operators σ_+ and σ_- to decompose mixed concepts.

Theorem 4. *Let $\mathbb{K} = (G, M, I)$ be a formal context and let $(A, B) \in \underline{\mathbb{B}}^{\#}(\mathbb{K})$. Then $(A, B) = \sigma_+(A, B) \wedge \sigma_-(A, B)$.*

Example 4. We continue with the formal context in Example 2. The mixed context presents an aggregate of 30 concepts. In Table 3, we show some of the concepts in $\underline{\mathbb{B}}^{\#}(\mathbb{K})$ together with their decomposition. To avoid massive listings, we have selected those concepts in which decomposition appear non-purely positive or negative sets $\sigma_+(A, B)$ and $\sigma_-(A, B)$, that is, such that $\mathrm{Neg}(\sigma_+(A, B)) \neq \varnothing$ or $\mathrm{Pos}(\sigma_-(A, B)) \neq \varnothing$.

Table 3. Decomposition of some of the concepts in $\underline{\mathbb{B}}^{\#}(\mathbb{K})$ of Example 1.

$(A, B) \in \underline{\mathbb{B}}^{\#}(\mathbb{K})$	$\sigma_+(A, B)$	$\sigma_-(A, B)$
$\{\mathrm{c}, \overline{\mathrm{b}}, \overline{\mathrm{d}}\}$	$\{\mathrm{c}\}$	$\{\mathrm{c}, \overline{\mathrm{b}}, \overline{\mathrm{d}}\}$
$\{\mathrm{c}, \overline{\mathrm{a}}, \overline{\mathrm{d}}\}$	$\{\mathrm{c}\}$	$\{\mathrm{c}, \overline{\mathrm{a}}, \overline{\mathrm{d}}\}$
$\{\mathrm{c}, \overline{\mathrm{a}}, \overline{\mathrm{b}}, \overline{\mathrm{d}}\}$	$\{\mathrm{c}\}$	$\{\mathrm{c}, \overline{\mathrm{a}}, \overline{\mathrm{b}}, \overline{\mathrm{d}}\}$
$\{\mathrm{b}, \mathrm{d}, \overline{\mathrm{a}}\}$	$\{\mathrm{b}, \mathrm{d}, \overline{\mathrm{a}}\}$	$\{\overline{\mathrm{a}}\}$
$\{\mathrm{b}, \mathrm{d}, \overline{\mathrm{a}}, \overline{\mathrm{c}}\}$	$\{\mathrm{b}, \mathrm{d}, \overline{\mathrm{a}}\}$	$\{\mathrm{b}, \mathrm{d}, \overline{\mathrm{a}}, \overline{\mathrm{c}}\}$
$\{\mathrm{b}, \mathrm{c}, \overline{\mathrm{a}}\}$	$\{\mathrm{b}, \mathrm{c}, \overline{\mathrm{a}}\}$	$\{\overline{\mathrm{a}}\}$
$\{\mathrm{b}, \mathrm{c}, \overline{\mathrm{a}}, \overline{\mathrm{d}}\}$	$\{\mathrm{b}, \mathrm{c}, \overline{\mathrm{a}}\}$	$\{\mathrm{c}, \overline{\mathrm{a}}, \overline{\mathrm{d}}\}$
$\{\mathrm{b}, \mathrm{c}, \mathrm{d}, \overline{\mathrm{a}}\}$	$\{\mathrm{b}, \mathrm{c}, \mathrm{d}, \overline{\mathrm{a}}\}$	$\{\overline{\mathrm{a}}\}$
$\{\mathrm{a}, \mathrm{d}, \overline{\mathrm{b}}, \overline{\mathrm{c}}\}$	$\{\mathrm{a}, \mathrm{d}, \overline{\mathrm{b}}, \overline{\mathrm{c}}\}$	$\{\mathrm{a}, \mathrm{d}, \overline{\mathrm{b}}, \overline{\mathrm{c}}\}$
$\{\mathrm{a}, \mathrm{c}, \overline{\mathrm{b}}, \overline{\mathrm{d}}\}$	$\{\mathrm{a}, \mathrm{c}, \overline{\mathrm{b}}, \overline{\mathrm{d}}\}$	$\{\mathrm{c}, \overline{\mathrm{b}}, \overline{\mathrm{d}}\}$
$\{\mathrm{a}, \mathrm{b}, \overline{\mathrm{c}}, \overline{\mathrm{d}}\}$	$\{\mathrm{a}, \mathrm{b}, \overline{\mathrm{c}}, \overline{\mathrm{d}}\}$	$\{\mathrm{a}, \mathrm{b}, \overline{\mathrm{c}}, \overline{\mathrm{d}}\}$
$\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \overline{\mathrm{a}}, \overline{\mathrm{b}}, \overline{\mathrm{c}}, \overline{\mathrm{d}}\}$	$\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \overline{\mathrm{a}}, \overline{\mathrm{b}}, \overline{\mathrm{c}}, \overline{\mathrm{d}}\}$	$\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}, \overline{\mathrm{a}}, \overline{\mathrm{b}}, \overline{\mathrm{c}}, \overline{\mathrm{d}}\}$

As a consequence of this theorem, and of the theoretical results of the previous section, we have the following decomposition result. This will help us provide an algorithm to compute $\underline{\mathbb{B}}^{\#}(\mathbb{K})$ from $\underline{\mathbb{B}}(\mathbb{K})$ and $\underline{\mathbb{B}}(\overline{\mathbb{K}})$.

Theorem 5. *Let $\mathbb{K} = (G, M, I)$ be a formal context and let $(A, B) \in \underline{\mathbb{B}}^{\#}(\mathbb{K})$. Then $(A_+, B_+) = \pi_+(A, B) \in \underline{\mathbb{B}}(\mathbb{K})$ and $(A_-, B_-) = \pi_-(A, B) \in \underline{\mathbb{B}}(\overline{\mathbb{K}})$ verify that $(A, B) = (A_+ \cap A_-, B_+ \cup B_-)$.*

Furthermore, we can say:

Theorem 6. *Let $\mathbb{K} = (G, M, I)$ be a formal context. The mappings:*

$$\begin{array}{rcl|rcl} \phi : \mathrm{Ext}(\mathbb{K}) \times \mathrm{Ext}(\overline{\mathbb{K}}) & \to & \mathrm{Ext}^{\#}(\mathbb{K}) & \psi : \mathrm{Int}(\mathbb{K}) \times \mathrm{Int}(\overline{\mathbb{K}}) & \to & \mathrm{Int}^{\#}(\mathbb{K}) \\ (A_1, A_2) & \mapsto & A_1 \cap A_2 & (B_1, B_2) & \mapsto & (B_1 \cup B_2)^{\Downarrow\Uparrow} \end{array}$$

are well defined and surjective.

The meaning of the last two theorems is that we can compute all intents of $\underline{\mathbb{B}}^{\#}(\mathbb{K})$ by traversing the intents of $\underline{\mathbb{B}}(\mathbb{K})$ and $\underline{\mathbb{B}}(\overline{\mathbb{K}})$, taking their union, and computing the closure. In fact, we can say that:

Corollary 3. *Let $\mathbb{K} = (G, M, I)$ be a formal context. Then*

$$\mathrm{Int}^{\#}(\mathbb{K}) = \left\{B = B_1 \cup B_2, B_1 \in \mathrm{Int}(\mathbb{K}), B_2 \in \mathrm{Int}(\overline{\mathbb{K}}),\ \textit{and } B = B^{\Downarrow\Uparrow}\right\}$$

Now, for the sake of readability, let us introduce the notation for the following mappings:

Notation 3. *Let $\Delta_{+}^{-} := \pi_{-} \circ e_{+} : \underline{\mathbb{B}}(\mathbb{K}) \to \underline{\mathbb{B}}(\overline{\mathbb{K}})$ and $\Delta_{-}^{+} := \pi_{+} \circ e_{-} : \underline{\mathbb{B}}(\overline{\mathbb{K}}) \to \underline{\mathbb{B}}(\mathbb{K})$.*

Lemma 3. *Consider a formal context $\mathbb{K} = (G, M, I)$:*

1. *If $(A, B) \in \underline{\mathbb{B}}(\mathbb{K})$ and we call $(C, D) = \Delta_{+}^{-}(A, B) = \pi_{-}(A, A^{\Uparrow}) \in \underline{\mathbb{B}}(\overline{\mathbb{K}})$, then, $(A, B \cup D) = e_{+}(A, B) \in \underline{\mathbb{B}}^{\#}(\mathbb{K})$.*
2. *If $(A, B) \in \underline{\mathbb{B}}(\overline{\mathbb{K}})$ and we call $(C, D) = \Delta_{-}^{+}(A, B) = \pi_{+}(A, A^{\Uparrow}) \in \underline{\mathbb{B}}(\mathbb{K})$, then, $(A, B \cup D) = e_{-}(A, B) \in \underline{\mathbb{B}}^{\#}(\mathbb{K})$.*

Thus, Δ_{+}^{-} and Δ_{-}^{+} can be used to find concepts in the other simple lattice that complete (by union) a concept in $\underline{\mathbb{B}}^{\#}(\mathbb{K})$. In addition, $e_{+}(A, B) = (A, B \cup D)$ is the greatest concept (in the sense of the order $\leq$ in $\underline{\mathbb{B}}^{\#}(\mathbb{K})$) such that its intent contains B: $e_{+}(A, B) = \sup\{(R, S) \in \underline{\mathbb{B}}^{\#}(\mathbb{K}) : B \subseteq S\}$, and analogously in $\underline{\mathbb{B}}(\overline{\mathbb{K}})$. This can be checked in Table 2, where we can see that the embeddings produce the intents in $\underline{\mathbb{B}}^{\#}(\mathbb{K})$ with the minimum cardinality containing B.

3.3 An Algorithm for Computing the Mixed Lattice

In this, section, we present an algorithm to compute the mixed concept lattice $\underline{\mathbb{B}}^{\#}(\mathbb{K})$ from the individual positive and negative lattices. It is based in the previous theoretical results about the decomposition of the intents of the mixed lattice. Furthermore, we will make use of other technical results to improve the performance of the algorithm:

Lemma 4 ([14]). *Let $\mathbb{K} = (G, M, I)$ be a formal context, and consider the associated mixed context. Then, the top concept of $\underline{\mathbb{B}}^{\#}(\mathbb{K})$ is $(G, G^{\Uparrow})$ and its bottom concept is $(\varnothing, M\overline{M})$.*

For the second result, we need to define the notion of *consistent* set.

Definition 1. *A set A is said to be* consistent *if $A \cap \overline{A} = \varnothing$ or, equivalently, if $\mathrm{Pos}(A) \cap \mathrm{Neg}(A) = \varnothing$, that is, if it does not contain a given attribute and its negation.*

Lemma 5 ([13]). *Let $\mathbb{K} = (G, M, I)$ be a formal context, and consider the associated mixed lattice $\underline{\mathbb{B}}^{\#}(\mathbb{K})$. The intent of all the concepts but $(\varnothing, M\overline{M})$ are consistent sets.*

These results allow us to propose the Algorithm 1. This algorithm iterates through all the concepts in the positive lattice $\underline{\mathbb{B}}(\mathbb{K})$, and for each concept (A, B) it computes the concept (C, D) given by Lemma 3, so that $(A, B \cup D)$ is indeed a mixed concept and is added to the result $\mathbb{L}$ (lines 3 − 4). Then, a queue $\mathcal{Q}$ is built consisting of all proper subconcepts of (C, D), excluding the bottom of $\underline{\mathbb{B}}(\overline{\mathbb{K}})$ (line 5). This way, we avoid duplicating concepts in the enumeration. After this, the algorithm picks one element (C_1, D_1) of the queue in each iteration, builds the following candidate intent B_* and its corresponding intent $X = A \cap C_1$ (lines 7 − 8). If $X = \varnothing$, this means that B_* (and its closure $B_*^{\Downarrow\Uparrow} = X^{\Uparrow}$) is not *consistent*, therefore they cannot be the intents of any concept in the mixed lattice, according to Lemma 5. In this case, all the subconcepts of (C_1, D_1) will verify this same condition, hence they are removed from the queue $\mathcal{Q}$ (lines 9 – 11). The last comprobation (lines 12 – 13) adds (X, B_*) to $\mathbb{L}$ only if B_* is closed in the mixed context. Finally, the algorithm returns $\mathbb{L} \cup \{\varnothing, M\overline{M}\}$, since this latter is the only known (Lemma 4) concept not included in the previous steps (line 14).

Algorithm 1: Compute Mixed Concepts

Input: $\underline{\mathbb{B}}(\mathbb{K})$: The lattice associated to $\mathbb{K}$. $\underline{\mathbb{B}}(\overline{\mathbb{K}})$: The lattice associated to $\overline{\mathbb{K}}$.
Output: $\underline{\mathbb{B}}^{\#}(\mathbb{K})$

1 $\mathbb{L} := \varnothing$
2 **for** $(A, B) \in \underline{\mathbb{B}}(\mathbb{K})$ **do**
3 $\quad (C, D) := \Delta_+^-(A, B) = \pi_-(A, A^{\Uparrow})$
4 $\quad \mathbb{L} \leftarrow \mathbb{L} \cup \{(A, B \cup D)\}$
5 $\quad \mathcal{Q} := \{(C_1, D_1) \in \underline{\mathbb{B}}(\overline{\mathbb{K}}) : \bot < (C_1, D_1) < (C, D)\}$
6 $\quad$ **for** $(C_1, D_1) \in \mathcal{Q}$ **do**
7 $\quad\quad B_* := B \cup D_1$
8 $\quad\quad X := B_*^{\Downarrow} = A \cap C_1$
9 $\quad\quad$ **if** $|X| = 0$ **then**
10 $\quad\quad\quad$ Remove from $\mathcal{Q}$ all subconcepts of (C_1, D_1)
11 $\quad\quad\quad$ Go to line #6
12 $\quad\quad$ **if** $B_* = X^{\Uparrow}$ **then**
13 $\quad\quad\quad \mathbb{L} \leftarrow \mathbb{L} \cup \{(X, B_*)\}$
14 **return** $\underline{\mathbb{B}}^{\#}(\mathbb{K}) = \mathbb{L} \cup \{(\varnothing, M\overline{M})\}$

Next, we sketch a single step of Algorithm 1, starting in a specific concept $(A, B) \in \underline{\mathbb{B}}(\mathbb{K})$, to show the procedure in more detail.

1. For $(A, B) = (\{o1, o2, o5, o7\}, \{b\})$, we compute $D = \varnothing$ using the operator Δ_{+}^{-}. Then $(A, B \cup D) = (A, B)$ is added to $\mathbb{L}$.
2. The queue is $\mathcal{Q} = \{\{\overline{\mathrm{d}}\}, \{\overline{\mathrm{c}}\}, \{\overline{\mathrm{c}}, \overline{\mathrm{d}}\}, \{\overline{\mathrm{b}}\}, \{\overline{\mathrm{b}}, \overline{\mathrm{d}}\}, \{\overline{\mathrm{b}}, \overline{\mathrm{c}}\}, \{\overline{\mathrm{a}}\}, \{\overline{\mathrm{a}}, \overline{\mathrm{d}}\},$ $\{\overline{\mathrm{a}}, \overline{\mathrm{c}}\}, \{\overline{\mathrm{a}}, \overline{\mathrm{b}}, \overline{\mathrm{d}}\}\}$.
3. We loop over the items in $\mathcal{Q}$:
   (a) For $D_1 = \{\overline{\mathrm{d}}\}$, we have $B_* = \{\mathrm{b}, \overline{\mathrm{d}}\}$ and $X = \{\mathrm{o2}, \mathrm{o7}\}$. Since $X^{\Uparrow} = B_*$, (X, B_*) is added to $\mathbb{L}$.
   (b) For $D_1 = \{\overline{\mathrm{c}}\}$, it is $B_* = \{\mathrm{b}, \overline{\mathrm{c}}\}$ and $X = \{\mathrm{o1}, \mathrm{o2}\}$. As before, (X, B) is added to $\mathbb{L}$.
   (c) For $D_1 = \{\overline{\mathrm{c}}, \overline{\mathrm{d}}\}$, we have $B_* = \{\mathrm{b}, \overline{\mathrm{c}}, \overline{\mathrm{d}}\}$ and $X = \{\mathrm{o2}\}$. This pair is rejected since B_* is not closed: $X^{\Uparrow} = \{\mathrm{a}, \mathrm{b}, \overline{\mathrm{c}}, \overline{\mathrm{d}}\} \neq B_*$.
   (d) For $D_1 = \{\overline{\mathrm{b}}\}$, we can compute $B_* = \{\mathrm{b}, \overline{\mathrm{b}}\}$ and then $X = \varnothing$. This means B_* is not consistent, so the algorithm removes $\{\{\overline{\mathrm{b}}\}, \{\overline{\mathrm{b}}, \overline{\mathrm{d}}\}, \{\overline{\mathrm{b}}, \overline{\mathrm{c}}\},$ $\{\overline{\mathrm{a}}, \overline{\mathrm{b}}, \overline{\mathrm{d}}\}\}$ from $\mathcal{Q}$.
   (e) For $D_1 = \{\overline{\mathrm{a}}\}$, we have $B_* = \{\mathrm{b}, \overline{\mathrm{a}}\}$ and $X = \{\mathrm{o1}, \mathrm{o5}, \mathrm{o7}\}$. B_* is closed so we add (X, B_*) to $\mathbb{L}$.
   (f) For $D_1 = \{\overline{\mathrm{a}}, \overline{\mathrm{d}}\}$, it is $B_* = \{\mathrm{b}, \overline{\mathrm{a}}, \overline{\mathrm{d}}\}$ and $X = \{\mathrm{o7}\}$, but B_* is not closed: $X^{\Uparrow} = \{\mathrm{b}, \mathrm{c}, \overline{\mathrm{a}}, \overline{\mathrm{d}}\} \neq B_*$, so (X, B_*) is rejected.
   (g) For $D_1 = \{\overline{\mathrm{a}}, \overline{\mathrm{c}}\}$, we have $B_* = \{\mathrm{b}, \overline{\mathrm{a}}, \overline{\mathrm{c}}\}$ and $X = \{\mathrm{o1}\}$. As before, B_* is not closed so (X, B_*) is rejected.

Note that, if $\underline{\mathbb{B}}(\mathbb{K})$ and $\mathcal{Q}$ are traversed in lectic order, then $\underline{\mathbb{B}}^{\#}(\mathbb{K})$ is also built in lectic order.

4 Conclusions and Future Work

The classical FCA paradigm studies the presence of an attribute for an object and does not consider the absence of the attribute as information explicitly. In this paper, we adopt the line of modelling both positive and negative information, and present a line of work on the relationship between the concept lattices of formal contexts consisting of only positive or negative information, and the concept lattice when the mixture of both types of information is considered.

This line of work has as its starting point a series of theoretical results presented in this article. These results tell us how the individual lattices map onto the mixed lattice, thanks to some *embedding* and *projection* operators that, jointly, form Galois connections between the $\underline{\mathbb{B}}(\mathbb{K})$ and $\underline{\mathbb{B}}(\overline{\mathbb{K}})$ and the mixed lattice $\underline{\mathbb{B}}^{\#}(\mathbb{K})$. Furthermore, we establish decomposition or representation results of the mixed concepts according to the closure operators induced by these Galois connections.

These decomposition results have allowed us to present a proposal of an algorithm for the computation of the concepts of the mixed lattice from the concepts of the individual lattices. Note that this algorithm requires to have precomputed $\mathbb{B}(\mathbb{K})$ and $\mathbb{B}(\overline{\mathbb{K}})$. But they can be computed in parallel.

As future work we aim to optimise this algorithm and exploit its *divide and conquer* strategy to build pyramid-like algorithms for the computation of massive concept lattices. Furthermore, the application of logic inference systems for mixed attributes may propose new strategies in this sense. In addition, we plan to apply all this results to the study of the minimal generators of the mixed lattice and of the implication bases of mixed contexts.

References

1. Cordero, P., Enciso, M., López, D., Mora, A.: A conversational recommender system for diagnosis using fuzzy rules. Expert Syst. Appl. **154**, 113449 (2020)
2. Cordero, P., Enciso, M., Mora, A., Ojeda-Aciego, M., Rossi, C.: Knowledge discovery in social networks by using a logic-based treatment of implications. Knowl. Based Syst. **87**, 16–25 (2015)
3. Cordero, P., Enciso, M., Mora, Á., Ojeda-Aciego, M., Rossi, C.: A formal concept analysis approach to cooperative conversational recommendation. Int. J. Comput. Intell. Syst. **13**, 1243–1252 (2020)
4. Ganter, B., Obiedkov, S.: More expressive variants of exploration. In: Conceptual Exploration, pp. 237–292. Springer, Heidelberg (2016). https://doi.org/10.1007/978-3-662-49291-8_6
5. Priss, U.: Diagrammatic representation of conceptual structures. In: Braud, A., Buzmakov, A., Hanika, T., Le Ber, F. (eds.) ICFCA 2021. LNCS (LNAI), vol. 12733, pp. 281–289. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-77867-5_19
6. Gasmi, G., Yahia, S.B., Nguifo, E.M., Bouker, S.: Extraction of association rules based on literalsets. In: Song, I.Y., Eder, J., Nguyen, T.M. (eds.) DaWaK 2007. LNCS, vol. 4654, pp. 293–302. Springer, Heidelberg (2007). https://doi.org/10.1007/978-3-540-74553-2_27
7. Ibrahim, M.H., Missaoui, R., Vaillancourt, J.: Identifying influential nodes in two-mode data networks using formal concept analysis. IEEE Access **9**, 159549–159565 (2021)
8. Konecny, J.: Attribute implications in L-concept analysis with positive and negative attributes: validity and properties of models. Int. J. Approximate Reasoning **120**, 203–215 (2020)
9. Messaoudi, A., Missaoui, R., Ibrahim, M.-H.: Detecting overlapping communities in two-mode data networks using formal concept analysis. Revue des Nouvelles Technologies de l'Information, Extraction et Gestion des connaissances, RNTI-E-, **35** 189–200 2019
10. Missaoui, R., Nourine, L., Renaud, Y.: Generating positive and negative exact rules using formal concept analysis: problems and solutions. In: Medina, R., Obiedkov, S. (eds.) ICFCA 2008. LNCS (LNAI), vol. 4933, pp. 169–181. Springer, Heidelberg (2008). https://doi.org/10.1007/978-3-540-78137-0_13
11. Missaoui, R., Nourine, L., Renaud, Y.: Computing implications with negation from a formal context. Fund. Inform. **115**(4), 357–375 (2012)
12. Rodríguez-Jiménez, J.M., Cordero, P., Enciso, M., Mora, A.: Negative attributes and implications in formal concept analysis. Procedia Comput. Sci. **31**, 758–765 (2014)

13. Rodríguez-Jiménez, J., Cordero, P., Enciso, M., Rudolph, S.: Concept lattices with negative information: a characterization theorem. Inf. Sci. **369**, 51–62 (2016)
14. Rodríguez-Jiménez, J.M., Cordero, P., Enciso, M., Mora, A.: Data mining algorithms to compute mixed concepts with negative attributes: an application to breast cancer data analysis. Math. Methods Appli. Sci. **39**(16), 4829–4845 (2016)
15. Wille, R.: Restructuring lattice theory: an approach based on hierarchies of concepts. Ordered Sets **83**, 445–470 (1982)