



Simplification logic for the management of unknown information

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ABSTRACT

This paper aims to contribute to the extension of classical Formal Concept Analysis (FCA), allowing the management of unknown information. In a preliminary paper, we define a new kind of attribute implications to represent the knowledge from the information currently available. The whole FCA framework has to be appropriately extended to manage unknown information. This paper introduces a new logic for reasoning with this kind of implications, which belongs to the family of logics with an underlying Simplification paradigm. Specifically, we introduce a new algebra, named weak dual Heyting Algebra, that allows us to extend the Simplification logic for these new implications. To provide a solid framework, we also prove its soundness and completeness and show the advantages of the Simplification paradigm. Finally, to allow further use of this extension of FCA in applications, an algorithm for automated reasoning, which is directly built from logic, is defined.

1. Introduction

Formal Concept Analysis (FCA) [1] has become a solid tool with a solid mathematical basis to deal with the complete data life-cycle: information storage, knowledge extraction and representation, symbolic manipulation, and, finally, reasoning and inference [2]. The information is structured straightforwardly using a binary relation I between a set of objects G and attributes M . The tern (G, M, I) is named formal context. FCA provides methods for extracting knowledge from formal contexts and represents this knowledge using two approaches [3,2]. One is the so-called *concept lattice*, which depicts closed sets of objects-attributes, named *formal concepts*, in a hierarchy graph. An alternative representation is given by *attribute implications*, more oriented to symbolic manipulation, and based on the notion of the if-then rule. Formal concepts describe maximal sets of objects that share common attributes. Implications, on the other hand, establish a relationship between sets of attributes.

In [3, page 17], Ganter and Wille interpret each element in the relation $(g, m) \in I$ as “the object g holds the property m ”, but it does not declare anything about the absence of information. For instance, in Fig. 1a, we affirm that object 1 has property a , but we do not affirm whether this object has or has not b or c . This view, called “*epistemic view*” in [4], does not only concern how the information, stored in formal contexts, is interpreted but also how we process it. The two approaches of the original FCA knowledge representation (concept lattice and implications) preserve the epistemic view. In both of them, only positive information is presented (through positive attributes), corresponding to the explicit *crosses* in the formal context (see Fig. 1a). In contrast, the blanks do not lead us to any knowledge.

This epistemic interpretation of the information stored in the formal context matches the usual meaning of unknown or missing information. For instance, when some customers do not declare their age or gender, we do not use it to manage their profiles further.

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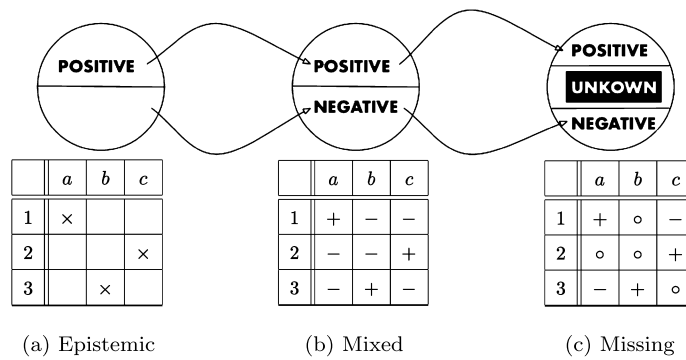


Fig. 1. Different views of the information stored in a formal context.

However, this view does not match the correct interpretation in some situations. For instance, the data regarding the consent for the use of different cookies (third-party cookies, functional cookies, session cookies, etc.) on several websites can be stored as a binary relation between a set of objects (URLs) and attributes (cookie purposes). An example of this view can be seen in [7]. In such a situation, a blank in the formal context must be interpreted as a negation of consent because you agree to the use of cookies only when you check the box. Then the site can store them (see Fig. 1b). For instance, this approach has been introduced in [8] where positive and negative truth values are considered. Other extensions of FCA consider negations but in a very different direction to the one discussed here. Thus, a negation at the level of concepts was defined in [9].

The first attempt to give a non-epistemic view of the context was to duplicate the attributes by considering each (positive) attribute and its negation [3]. As Missaoui et al. showed in [10], this approach tends to be inefficient because it generates redundant information. Although negative and positive information is complementary and mutually exclusive, this situation is not taken into account. For instance, a formal context could have the attributes “session_cookie_on” and “session_cookie_off” (opposite elements), and the implication “session_cookie_on implies not_session_cookie_off” must be preserved in the context. Both attributes are considered independent in this approach, and this relationship is not taken into account in the knowledge extraction and reasoning methods.

These relationships could reduce the specification and improve the performance of FCA methods by making the most of the semantics behind these negations. Several papers have addressed this issue [10–12]. For instance, FCA has been extended to deal with the so-called *mixed* (positive and negative) information by introducing the mixed concepts [13] and characterizing the lattice of such concepts. Moreover, in [14], methods to build the mixed concept lattice corresponding to a mixed formal context are explored. Those methods take advantage of the relationship between positive and negative information. A logic for reasoning about implications with mixed attributes was introduced in [15].

Therefore, a formal context can be interpreted under the epistemic approach (Fig. 1a) or the mixed view (Fig. 1b). In this work, we intend to develop the framework to reason with incomplete (unknown) information (Fig. 1c). This argument has been backed in [16–18], and is supported by real-life situations that substantiate the need to work with unknown information (for instance, when we haven’t got the information or when it does not make sense to speak about having, or not, an attribute). This situation is mentioned in [19]: *in real-life applications, missing data often appears for various reasons*. And these reasons are also mentioned in [20]: *incomplete formal contexts, conflict situations, and other similar cases*.

In [21], the authors present another point of view about what the unknown could mean. When there is a “large” amount of data in a formal context, it would be desirable to reduce the number of objects to handle them more efficiently. For this purpose, sets of objects can be compacted into a single object under the condition that the new object will have an attribute if all the original ones have it, will not have it if none of them have it, and will be unknown if some objects in the set have it and others do not. It is another example where unknown information appears.

In [17], a summary of several approaches to the handling of incomplete knowledge in FCA can be found. Furthermore, recent papers deal with partial formal contexts: in [4] the authors deal with *unknown or incomplete information*, a three-way concept analysis is introduced by [22], in [23] the authors use FCA to deal with incomplete formal contexts in conflict analysis, in [24] the authors built the concept lattice to optimize the selection of tags assigned to open data when part of metadata is missing, and in [25] the authors combine FCA with Neural Networks to analyze customers opinions.

In [26], we can see different ways of working with unknown information. The first difference lies in how we consider it. In our case, we consider as unknown information those cases where we have no information about the veracity of the attribute or we are not sure about it. Another point to consider is how we order the possible values: the values True, False, and Unknown. There are many cases where the order to take into account is the order of truthfulness, which considers true at the top, false at the bottom and unknown in the middle. This is not our case; as we have the uncertainty of not having any information we use an order of knowledge that considers the unknown as the bottom, and true and false are incomparable. Both show us the same information: I have information that is true, and I have information that is false.

This work is the sequel to [27] where we established a new theoretical framework to manage unknown information. We presented an Armstrong-like logic that is more oriented to describing the semantics of the implications. In the current paper, we approach the simplification paradigm for reasoning with implications [28], which emerges as a more application-oriented alternative to

Armstrong’s Axioms. In particular, it is more appropriate for designing automatic reasoning methods [29]. In this research line, we remark on two recent articles:

On the one hand, in the recent paper [4] the authors remain in the classical logic, considering a bi-valued approach enriching the usual FCA implication logic with the disjunction and assuming a set of operators inspired by the modal logic operators. In this way, they provide high expressive power with a classical framework. However, greater expressive power implies more complex methods of reasoning. Thus, the authors preserve the syntax and semantics of the classical logic as other authors did in the past [5,6], but they add a meta-level with some modal-like operators, impacting the complexity of the reasoning methods. A preliminary and similar approach, where also modal logic was the key to dealing with incomplete information, was proposed in [18].

On the other hand, Fuzzy Set Theory also addresses this problem. There exist some fuzzy extensions of FCA that consider positive and negative information. For instance, in [30], the author manages a very rich wide range of truth values with a solid algebraic structure, introducing a notion of implication for such structure. Although this approach to the missing information could be considered a good line of work because of the use of a fuzzy underlying framework, the full development of the whole theory is not trivial. It can be viewed as an open problem. Other important papers which stated the use of Fuzzy Logic in this topic in the past could be found in [31] or in [32]. In the middle term, in [33], the authors propose an extension of FCA by using Atanassov’s L-fuzzy sets as truthfulness values.

In [27] we took the articles [4,30] as inspirations, although our goal was to balance the expressive power with the possibility of designing efficient automated methods for implications. The design of those methods is the objective of the current paper.

The paper is structured as follows: Section 2 presents some preliminary notions and results that were introduced in [27]. In Section 3, we introduce the notion of weak dual Heyting algebra, which is our target algebra; we also prove that the main algebra presented in the preliminaries is a weak dual Heyting algebra. Section 4 presents two simplification logics that are sound and complete for the weak implications shown in the preliminaries. In Section 5, we present the paradigm of the simplification logic, that is, the rules of equivalence that hold with weak implications. Section 6 presents an algorithm to get the closure of the elements for a given set of weak implications. Finally, we present conclusions and further works in Section 7.

2. Preliminaries

As we have said in the above section, this work is the sequel of [27]. To ease the reading of this paper, we summarize the main results of that work.

2.1. The algebraic framework for unknown information

The 3-valued logics are usually conceived as an extension of the classical logics by adding an intermediate value to the truth-value set $\{F, T\}$. This value enriches the expressive power and induces an order considering a “borderline” value to be located between the other two values. This is the case of Łukasiewicz logic [34] or Kleene logic [35]. The main difference between these logics is the meaning of the third value, specifically the meaning of the negation of this value. In some sense, fuzzy logic also follows this idea. It can be considered a generalization since it introduces a set of (potentially infinite) values between the two truth values.

The usual interpretation of the truth value F is “totally false”, and the T corresponds to “totally true”. Then, the borderline can be seen as an intermediate knowledge between these two situations.

In our work, we interpret T as “we have information that it is true” and F as “we have information that it is false”. Then, the third truth value can be seen as “unknown”, i.e. we do not have any information about its truthfulness. For instance, in a health information system, the variable “being pregnant” matches this interpretation since the unknown value collects the situation where we don’t have information at all (perhaps we have not yet carried out a test or we don’t have enough privileges to check it).

Hence, we follow the line successfully adopted by some popular disciplines as relational databases [36]. An exhaustive review of this issue can be seen in [26].

We consider the \wedge -semilattice $\underline{\mathbf{3}} = (\mathbf{3}, \leq)$ where $\mathbf{3} = \{+, -, \circ\}$ and \leq is the reflexive closure of $\{(\circ, +), (\circ, -)\}$ (see Fig. 2a). Given a universal set U , a 3-set in U is a mapping $A : U \rightarrow \mathbf{3}$ where, for each $u \in U$, $A(u)$ represents the knowledge about the membership of u to A . Thus, $+$ means that u belongs to A (we call it positive information), $-$ means that u does not belong to A (we call it negative information), and \circ denotes the absence of information about the membership of u (which is called unknown information). As usual, the set of 3-sets on U inherits the \wedge -semilattice structure, denoted by $\underline{\mathbf{3}}^U = (\mathbf{3}^U, \sqsubseteq)$, by considering the point-wise ordering:

$$A \sqsubseteq B \text{ iff } A(u) \leq B(u) \text{ for all } u \in U.$$

Given a 3-set A , the support of A is the set $\text{Spp}(A) = \{u \in U \mid A(u) \neq \circ\}$. We define also the mappings $\text{Neg}, \text{Pos}, \text{Unk} : \mathbf{3}^U \rightarrow \mathcal{P}(U)$ as follows: for each $A \in \mathbf{3}^U$:

$$\text{Neg}(A) = \{u \in U \mid A(u) = -\} = A^{-1}(-)$$

$$\text{Pos}(A) = \{u \in U \mid A(u) = +\} = A^{-1}(+)$$

$$\text{Unk}(A) = \{u \in U \mid A(u) = \circ\} = A^{-1}(\circ)$$

An equivalent formalization of 3-sets can be found in [26], where only the known information (Positive or Negative) is given. It is represented as the so-called orthopairs using sets (P, N) with $P \cap N = \emptyset$.

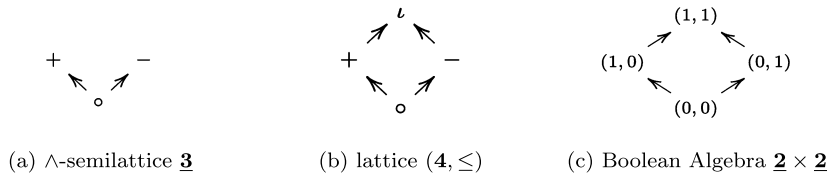


Fig. 2. Truthfulness's values.

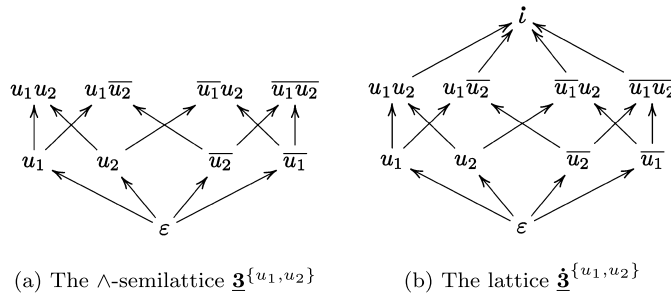


Fig. 3. Lattices from the set $\{u_1, u_2\}$.

When the support of a $\mathbf{3}$ -set A is finite, we express it as a sequence of elements (with no delimiters). It can be seen as an ordered re-writing of the concatenation of the elements of the orthopair, where the atoms in N are capped. For instance, $A = u_1 \bar{u}_5 u_7$ means that $A(u_1) = A(u_7) = +$, $A(u_5) = -$, and $A(x) = \circ$ in other cases. In particular, when $\text{Spp}(A) = \emptyset$, we denote $A = \varepsilon$ (i.e. the empty chain).

As aforementioned, we conceive the sets valued in $\mathbf{3}$ as the knowledge we have about one object's properties. Thus, we have a conjunctive interpretation of these sets, and consequently, when we join two different $\mathbf{3}$ -sets, we can find inconsistencies; that is, a property can be positive in one of the sets and negative in the other set. Then, in the joined set, we have an inconsistent element. The sets $\mathbf{3}$ and $\mathbf{3}^U$ are extended to model this situation. Hence, we introduce a fourth element, denoted i representing inconsistent or contradictory information. This element is going to be the maximum element and completes $\mathbf{3}$, transforming it into a lattice. This lattice $(\mathbf{4}, \leq)$ is shown in Fig. 2b and is isomorphic to the Boolean algebra $\underline{\mathbf{2}} \times \underline{\mathbf{2}}$ (see Fig. 2c). This algebraic structure is known as “information ordering” in the bi-lattices structure introduced by Belnap [37].

In addition, $\mathbf{4}^U$ denotes the set of mappings $A : U \rightarrow \mathbf{4}$ or $\mathbf{4}$ -sets, and we assume that $\mathbf{3}^U$ is a subset of $\mathbf{4}^U$. In the same way that we did for $\mathbf{3}^U$, the order of $\mathbf{4}$ is pointwise extended to $\mathbf{4}^U$. The $\mathbf{4}^U$ -sets can be seen as paraconsistent orthopairs [38], where the condition $P \cap N = \emptyset$ is omitted. Notice that the infimum of $(\mathbf{4}^U, \leq)$ is ε and the supremum is the $\mathbf{4}$ -set that maps any $u \in U$ to i , which is named *oxymoron* and denoted by i .

However, as in the classical propositional logic, from the semantics of the logic we introduce, when any contradiction appears, we can derive anything. Thus, we identify all paraconsistent orthopairs and use i as the representative element of this class. To formalize it, we define the following closure operator:

$$\mathcal{O} : \mathbf{4}^U \rightarrow \mathbf{4}^U \text{ being } \mathcal{O}(A) = \begin{cases} A & \text{if } A \in \mathbf{3}^U, \\ i & \text{otherwise.} \end{cases}$$

We denote by $\mathbf{3}^U$ its codomain $\mathcal{O}(\mathbf{4}^U) = \mathbf{3}^U \cup \{i\}$, which is a closure system in $(\mathbf{4}^U, \leq)$ and, therefore, it is a \wedge -subsemilattice of $(\mathbf{4}^U, \leq)$ (but not a sublattice) and $(\mathbf{3}^U, \sqsubseteq)$ is a complete lattice (see Fig. 3). Since both infima coincide, they will be denoted by the same symbol: \wedge . However, the supremum in $(\mathbf{4}^U, \leq)$ is denoted by the \vee , whereas in $(\mathbf{3}^U, \sqsubseteq)$ is denoted by \sqcup . Thus, we have that, for all $\{A_j : j \in J\} \subseteq \mathbf{3}^U$,

$$\bigsqcup_{j \in J} A_j = \mathcal{O}\left(\bigvee_{j \in J} A_j\right)$$

and, in particular,

$$\bigsqcup_{j \in J} A_j \neq i \quad \text{implies} \quad \bigvee_{j \in J} A_j = \bigvee_{j \in J} A_j. \tag{1}$$

Therefore, in $\mathbf{3}^U$, we have the set of consistent orthopairs belonging to $\mathbf{3}^U$ and the oxymoron i . The maximal sets of $\mathbf{3}^U$ are named the full sets, and the set of all of them is denoted by $\mathcal{F}\text{ull}(U)$. They are the super-atoms of $(\mathbf{3}^U, \sqsubseteq)$ and coincide with those orthopairs (P, N) such that $P \cup N = U$ and $P \cap N = \emptyset$. On the other hand, the atoms of $\mathbf{3}^U$, named singletons, are those $\mathbf{3}$ -sets whose support is a singleton.

We extend the mappings $\text{Neg}, \text{Pos}, \text{Unk} : \mathbf{3}^U \rightarrow \mathcal{P}(U)$ by considering $\text{Pos}(i) = \text{Neg}(i) = U$, $\text{Unk}(i) = \emptyset$.

\mathbb{P}	m_1	m_2	m_3
g_1	+	o	–
g_2	o	o	+
g_3	–	–	o

Fig. 4. Partial formal context \mathbb{P} .

Proposition 1. *The following assertions hold for all $A, B \in \mathfrak{3}^U$:*

1. $A \sqsubseteq B$ if and only if $\text{Neg}(A) \subseteq \text{Neg}(B)$ and $\text{Pos}(A) \subseteq \text{Pos}(B)$.
2. $A \in \mathcal{F}\text{ull}(U)$ if and only if $\text{Unk}(A) = \emptyset$, or equivalently $\text{Pos}(A) \cup \text{Neg}(A) = U$ and $\text{Pos}(A) \cap \text{Neg}(A) = \emptyset$.
3. The mappings Pos and Neg restricted to $\mathcal{F}\text{ull}(U)$ are bijections in $\mathcal{P}(U)$.

Finally, we define the operation $\bar{(\)} : \mathfrak{3}^U \rightarrow \mathfrak{3}^U$, which we call *opposite*, such that $\bar{i} = i$ and, for all $A \in \mathfrak{3}^U$, and $u \in U$,

$$\bar{A}(u) = \begin{cases} + & \text{if } A(u) = - \\ - & \text{if } A(u) = + \\ o & \text{if } A(u) = o \end{cases}$$

Thus, $\text{Neg}(\bar{A}) = \text{Pos}(A)$, $\text{Pos}(\bar{A}) = \text{Neg}(A)$, and $\text{Unk}(\bar{A}) = \text{Unk}(A)$.

2.2. Formal concept analysis for unknown information

We start the extension of the FCA framework by presenting the notion of *partial formal context*, introduced by Ganter in [21], which is defined as a triple $IP = (G, M, I)$ being G and M non-empty sets and $I : G \times M \rightarrow \mathfrak{3}$. We call the elements of G and M *objects* and *attributes* respectively. Given $g \in G$ and $m \in M$, the assignment $I(g, m) = +$ means that it is known that the object g has the attribute m ; $I(g, m) = -$ means that it is known that the object g has not the attribute m ; and $I(g, m) = o$ means that it is not known whether the object g has the attribute m or not. As in the classical case, these contexts are shown as tables (see Fig. 4, for instance).

The classical derivation operator is generalized by defining $(\)^\uparrow : \mathcal{P}(G) \rightarrow \mathfrak{3}^M$ and $(\)^\downarrow : \mathfrak{3}^M \rightarrow \mathcal{P}(G)$ as

$$X^\uparrow = \bigwedge_{g \in X} I(g, \), \quad \text{and} \quad Y^\downarrow = \{g \in G \mid Y(m) = I(g, m), \forall m \in \text{Spp}(Y)\}$$

A pair $(A, B) \in \mathcal{P}(G) \times \mathfrak{3}^M$ is said to be a (*formal*) *concept* if $A^\uparrow = B$ and $B^\downarrow = A$. As in the classical case, the concepts can be hierarchically ordered as $(A_1, B_1) \leq (A_2, B_2)$ if and only if $A_1 \subseteq A_2$ (or, equivalently, iff $B_2 \subseteq B_1$). Both operators form a Galois connection between the lattices $(\mathcal{P}(G), \subseteq)$ and $(\mathfrak{3}^M, \supseteq)$ and, therefore, concepts are fixed points of the Galois connection, and the set of all concepts with the hierarchical order constitutes a complete lattice.

In [27] we show that a classic formal context can be seen as a partial formal context, and we call a partial formal context without any unknown information a *total formal context*. We also proved the existence of a classic formal context whose formal concepts are isomorphic to the concepts of our partial formal context. Conversely, given a formal context, we proved the existence of a partial formal context whose concepts are isomorphic to the formal concepts of the classic formal context.

An expression $A \rightsquigarrow B$ with $A, B \in \mathfrak{3}^M$ is called *weak implication* (of attributes), and the set of weak implications is denoted by \mathcal{L}_M . Over this set, which we contemplate to be the language of the logic, we introduce the semantics as follows: We say that $C \in \mathfrak{3}^M$ is model of a weak implication $A \rightsquigarrow B \in \mathcal{L}_M$ if it satisfies that $A \subseteq C$ implies $B \subseteq C$. We denote it by $C \vDash A \rightsquigarrow B$.

By extension, we say that a partial formal context $IP = (G, M, I)$ is model of a weak implication $A \rightsquigarrow B \in \mathcal{L}_M$, or that $A \rightsquigarrow B$ is satisfied in IP , if $\{g\}^\uparrow \vDash A \rightsquigarrow B$ for all $g \in G$, and it is denoted by $IP \vDash A \rightsquigarrow B$. Moreover, IP is model of a set $\Sigma \subseteq \mathcal{L}_M$, denoted by $IP \vDash \Sigma$, if $IP \vDash A \rightsquigarrow B$ for all $A \rightsquigarrow B \in \Sigma$.

As in the classic case, by the properties of Galois connections, we have that

$$IP \vDash A \rightsquigarrow B \quad \text{iff} \quad A^\downarrow \subseteq B^\downarrow \quad \text{iff} \quad B \subseteq A^{\downarrow\uparrow}.$$

Now, we are ready to introduce the notion of *semantic entailment*: we say that $A \rightsquigarrow B \in \mathcal{L}_M$ is *semantically entailed* from $\Sigma \subseteq \mathcal{L}_M$, denoted by $\Sigma \vDash A \rightsquigarrow B$, when, for all partial formal contexts IP , we have that $IP \vDash \Sigma$ implies $IP \vDash A \rightsquigarrow B$.

To better understand our contribution, it is important to show the differences between our logic and propositional logic. Thus, our logic allows us to express that one attribute is unknown, while in the propositional logic, it is not possible since the models are valued in $\{0, 1\}$, where 0 means that the attribute is not true and 1 means that the attribute is true. Also, notice that, in propositional logic, given two propositional symbols p and q , the models of the implication $p \rightarrow q$ are those valuations of the propositional symbols that assign 0 to p or assign 1 to q , that is, the formula is interpreted as “ p is false, or q is true”. However, in the logic presented here, given two attributes a and b , a $\mathfrak{3}$ -set C is a model of the implication $a \rightsquigarrow b$ if either $C(a) = o$ or $C(a) = -$ or $C(b) = +$, i.e., the formula is interpreted as “either a is unknown, or it is known that a is false or that b is true”. In addition, the two semantic entailments also

differ. For instance, in propositional logic, $p \rightarrow q$ is equivalent to $\neg q \rightarrow \neg p$; that is, each of these formulas is semantically entailed from the other. However, in our logic, the implications $a \rightsquigarrow b$ and $\overline{b} \rightsquigarrow \overline{a}$ are not equivalent due to, for instance, the set $C = \overline{b}$ is a model for the first one but not for the second one because a is unknown in C .

Regarding the syntactic management to reason with weak implications, in [27] we introduced an axiomatic system following the classical paradigm of Armstrong’s Axioms [39]. Indeed, the syntactic formalization is the same as the original Armstrong’s paradigm, built on the new enriched underlying semantics. This axiomatic system, that we will name Armstrong-style Axiomatic System, is $\{[Inc], [Augm], [Trans]\}$, where $[Inc]$, $[Augm]$ and $[Trans]$ are defined as follows:

- $[Inc]$: infer $A \sqcup B \rightsquigarrow A$;
- $[Augm]$: from $A \rightsquigarrow B$ infer $A \sqcup C \rightsquigarrow B \sqcup C$;
- $[Trans]$: from $A \rightsquigarrow B$ and $B \rightsquigarrow C$ infer $A \rightsquigarrow C$;

where A, B, C are $\mathfrak{3}^M$ -sets. These rules are known as Inclusion, Augmentation, and Transitivity, respectively. Finally in [27], we proved that this axiomatic system is sound and complete.

3. Weak dual Heyting algebra

Classical Simplification Logic [29] lies in the Boolean algebra of sets. However, it has been extended to the fuzzy framework by [40] using a weaker structure, the complete dual Heyting algebra, which is a complete lattice endowed with a difference operation \setminus that satisfies the following adjoint property:

$$a \leq b \vee c \quad \text{if and only if} \quad a \setminus b \leq c \tag{2}$$

In [40] it was also proved that the following property holds in any complete dual Heyting algebra:

$$a \vee ((a \vee b) \setminus c) = a \vee (b \setminus c) \tag{3}$$

Now, in this work, we show it is possible to generalize the Simplification Logic to work with unknown information. In particular, we prove that neither a Boole algebra as in [29] nor a dual Heyting Algebra as in [40], is necessary. We also build a Simplification-style logic on an underlying algebra with just some of the properties of the dual Heyting algebra, like the following:

Definition 2. An algebra $(L, \vee, \wedge, \setminus, \perp, \top)$ is said to be a weak dual Heyting algebra, if $(L, \vee, \wedge, \perp, \top)$ is a complete lattice and one has that:

- $[wH1]$. $a \vee b \neq \top$ implies $(a \vee b) \setminus a \leq b$ for all $a, b \in L$.
- $[wH2]$. $a \setminus b \leq a$ for all $a, b \in L$.
- $[wH3]$. $a \setminus b = \perp$ if and only if $a \leq b$ for all $a, b \in L$.
- $[wH4]$. $a \vee b = a \vee (b \setminus a)$ for all $a, b \in L$.

To provide another characterization of weak dual Heyting algebras, in the following proposition we introduce an alternative of items $[wH3]$ and $[wH4]$. Its proof checks that, when $[wH1]$ and $[wH2]$ hold, items $[wH3]$ and $[wH4]$ are equivalent to the two new items presented.

Proposition 3. Let $(L, \vee, \wedge, \setminus, \perp, \top)$ be a complete lattice. The algebra $(L, \vee, \wedge, \setminus, \perp, \top)$ is a weak dual Heyting algebra, if and only if it satisfies $[wH1]$, $[wH2]$ and one has that:

- $[wH3']$. $a \leq b$ implies $a \setminus b = \perp$ for all $a, b \in L$.
- $[wH4']$. $b \leq a \vee (b \setminus a)$ for all $a, b \in L$.

Proof. It is straightforward that $[wH3']$ and $[wH4']$ holds in any weak dual Heyting algebra. Now, assume that $[wH2]$, $[wH3']$, and $[wH4']$ hold and prove $[wH3]$ and $[wH4]$. If $a \setminus b = \perp$, by $[wH4']$, we have that $a \leq b \vee (a \setminus b) = b \vee \perp = b$. The converse implication is due to $[wH3']$. Finally, by $[wH4']$ and $[wH2]$, we have that $a \vee b \leq a \vee (b \setminus a) \leq a \vee b$. \square

An interesting result related to the structure of weak dual Heyting algebras is shown in the following proposition. It highlights a necessary property to check whether an algebra is a weak Heyting dual algebra.

Proposition 4. Let $(L, \vee, \wedge, \setminus, \perp, \top)$ be a weak dual Heyting algebra. Then $a \setminus \perp = a$ for all $a \in L$.

Proof. Since \perp is the infimum of L and, by using $[wH4]$, we have that, for all $a \in L$, $a \setminus \perp = \perp \vee (a \setminus \perp) = \perp \vee a = a$. \square

We conclude the study of the algebra introduced here with the following proposition, whose proof is straightforward. It establishes the relationship between the complete dual Heyting algebras and the weak dual Heyting algebras.

Proposition 5. Any complete dual Heyting algebra is a weak dual Heyting algebra.

Now, we introduce two complete dual Heyting algebras, denoted by $\underline{4}$ and $\underline{4}^U$ respectively, from which we build the weak dual Heyting algebra $\underline{3}^U$ that will be the key point to define the Simplification Logic. This structure $\underline{3}^U$ shows that Proposition 5 can not be improved, i.e., weak dual algebras are not complete dual Heyting algebras.

The complete dual Heyting algebra $\underline{4}$ is $(\underline{4}, \vee, \wedge, \setminus, \circ, i)$ where $(\underline{4}, \vee, \wedge, \setminus, \circ, i)$ is the complete lattice defined by \leq and $\setminus : \underline{4} \times \underline{4} \rightarrow \underline{4}$ is given by the following table:

\setminus	\circ	$+$	$-$	i
\circ	\circ	\circ	\circ	\circ
$+$	$+$	\circ	$+$	\circ
$-$	$-$	$-$	\circ	\circ
i	i	$-$	$+$	\circ

The complete dual Heyting algebra $\underline{4}^U$ is defined by pointwise extending the previous one to $\underline{4}^U$, i.e. $\underline{4}^U = (\underline{4}^U, \vee, \wedge, \setminus, \varepsilon, i)$ where $(X \setminus Y)(u) = X(u) \setminus Y(u)$, for all $u \in U$.

The following straightforward proposition shows that the restriction of the difference operation to $\underline{3}$ can be considered an operation in $\underline{3}$.

Proposition 6. If $X, Y \in \underline{3}^U$, then $X \setminus Y \in \underline{3}^U$. In particular,

1. $X \setminus i = \varepsilon$ for all $X \in \underline{3}^U$.
2. If $Y \in \mathcal{F}ull(U)$ then $i \setminus Y = \overline{Y} \in \mathcal{F}ull(U)$.
3. If $Y \in \underline{3}^U \setminus \mathcal{F}ull(U)$ then $i \setminus Y = i$.
4. If $X, Y \in \underline{3}^U$ then $X \setminus Y \in \underline{3}^U$ and $X \setminus Y = \mathcal{O}(X \setminus Y)$.

The implication stated in Proposition 5 is not true in the reverse direction. A counterexample is $\underline{3}^U = (\underline{3}^U, \sqcup, \wedge, \setminus, \varepsilon, i)$. To show this situation, we first prove that this structure is a weak dual Heyting algebra, and later we show, with a particular case, that it is not a complete dual Heyting algebra.

Proposition 7. The structure $\underline{3}^U = (\underline{3}^U, \sqcup, \wedge, \setminus, \varepsilon, i)$ is a weak dual Heyting algebra.

Proof. As has been shown before, $(\underline{3}^U, \sqcup, \wedge, \varepsilon, i)$ is a complete lattice and, by Proposition 6, the set $\underline{3}^U$ is closed for the operation \setminus defined in $\underline{4}^U$. Thus, we prove that the properties given in the Definition 2 hold.

First, due we have that $X \sqcup Y \neq i$ implies $X \sqcup Y = X \vee Y$ and, since $\underline{4}^U$ is a complete dual Heyting algebra, one has $(X \vee Y) \setminus X \sqsubseteq Y$.

Second, for all $X, Y \in \underline{3}^U$, we have $X \setminus Y \sqsubseteq X$, and also that $X \setminus Y = \varepsilon$ if and only if $X \sqsubseteq Y$. The proof is straightforward from Proposition 6 if $X = i$ or $Y = i$. Otherwise, let $X, Y \in \underline{3}^U$. From item 4 of Proposition 6, $X \setminus Y \in \underline{4}^U$. Since $\underline{4}^U$ is a complete dual Heyting algebra, we have that $X \setminus Y \sqsubseteq X$, and $X \setminus Y = \varepsilon$ if and only if $X \sqsubseteq Y$.

Third prove $X \sqcup Y = X \sqcup (Y \setminus X)$ we have two situations, namely $X \sqcup Y \neq i$ or $X \sqcup Y = i$. In the first one, the proof is straightforward from the fact that $\underline{4}^U$ is a complete dual Heyting algebra. In the second situation, we distinguish the following cases. If either $X = i$ or $Y = i$ then we have that $X \sqcup (Y \setminus X) = i = X \sqcup Y$. Otherwise, we have that there exists an element $a \in X$ such that $\overline{a} \in Y$. Then $\overline{a} \in Y \setminus X$ and $X \sqcup (Y \setminus X) = i = X \sqcup Y$. \square

Before introducing the counterexample mentioned above, we present the following technical result, which will be helpful throughout the paper.

Proposition 8. Let $X, Y, Z \in \underline{3}^M$, the following assertions are fulfilled:

1. $X \sqcup Y = i$ if and only if $\overline{X} \wedge Y \neq \varepsilon$.
2. If $X \sqcup (Z \setminus Y) \neq i$ and $X \sqcup Z = i$, then $X \sqcup Y = i$.
3. If $X \sqcup ((X \sqcup Z) \setminus Y) \neq i$ then $X \sqcup ((X \sqcup Z) \setminus Y) = X \sqcup (Z \setminus Y)$
4. If $X \neq i$ then $Y \wedge (X \setminus Y) = \varepsilon$.

Proof. Item 1 is straightforward from the definition.

For item 2, assume that $X \sqcup (Z \setminus Y) \neq i$ and $X \sqcup Z = i$. From item 1, we have that $\overline{X} \wedge (Z \setminus Y) = \varepsilon$ and $\overline{X} \wedge Z \neq \varepsilon$. Therefore, $\overline{X} \wedge Y \neq \varepsilon$ or, equivalently, from item 1, $X \sqcup Y = i$.

Let us prove item 3. Assume $X \sqcup ((X \sqcup Z) \setminus Y) \neq i$. Since $\underline{4}^U$ is a complete dual Heyting algebra, from (1) and (3), we have that:

$$X \sqcup ((X \sqcup Z) \setminus Y) = X \vee ((X \vee Z) \setminus Y) = X \vee (Z \setminus Y) = X \sqcup (Z \setminus Y).$$

Finally, let $X \in 3^U$ and $Y \in 3^U$. If $Y = i$ we have that $X \setminus Y = \varepsilon$ and consequently $Y \wedge (X \setminus Y) = \varepsilon$. Otherwise ($Y \neq i$), let's denote $Z = Y \wedge (X \setminus Y)$ and we prove that $Z = \varepsilon$, i.e. $Z(u) = \circ$ for all $u \in U$. Given $u \in U$, if $X(u) = \circ$ or $Y(u) = \circ$ or $X(u) = Y(u)$ we have straightforward that $Z(u) = \circ$. If $X(u) \neq Y(u)$, $X(u) \neq \circ$ and $Y(u) \neq \circ$ we have that either $X(u) = +$ and $Y(u) = -$, or $X(u) = -$ and $Y(u) = +$. Thus, in all the situations, we have that $Z(u) = Y(u) \wedge X(u) = \circ$. \square

We conclude this section with an example that shows that $\underline{3}^U$ is not necessarily a complete dual Heyting algebra.

Example 1. Given $U = \{a, b\}$, the algebra $\underline{3}^U$ is not a complete dual Heyting algebra because it does not satisfy the adjoint property (2). For instance, for $X = ab$ and $Y = \overline{ab}$, we have that $i \sqsubseteq X \sqcup Y$ but $i \setminus X = \overline{ab} \not\sqsubseteq Y$.

4. Axiomatic systems for weak implications

As stated in the introduction, the main objective of this work is to provide a logic based on the Simplification paradigm [29] for reasoning with implications. Thus, we present two approaches by introducing two axiomatic systems for weak implications and prove that both axiomatic systems and the one introduced [27] are equivalent.

Definition 9. *Simplification Axiomatic System* is $\{[Inc], [Key], [Simp]\}$ where $[Inc]$, $[Key]$ and $[Simp]$ are, respectively, defined as follows:

Inclusion: Infer $A \sqcup B \rightsquigarrow A$,

Key: From $A \rightsquigarrow b$ infer $A \sqcup \overline{b} \rightsquigarrow i$,

Simplification: From $A \rightsquigarrow B$ and $C \rightsquigarrow D$ infer $A \sqcup (C \setminus B) \rightsquigarrow D$,

for all $A, B, C \in \underline{3}^M$ and all singletons $b \in 3^M$.

Our second proposal replaces the “Key” inference rule with a version of the classical union rule.

Definition 10. *U-Simplification Axiomatic System* is $\{[Inc], [Simp], [Un]\}$ where $[Un]$ is defined as follows: for all 3^M -sets A, B, C ,

Union: From $A \rightsquigarrow B$ and $A \rightsquigarrow C$ infer $A \rightsquigarrow B \sqcup C$.

The notion of syntactic derivation (for the three axiomatic systems) is introduced in the standard way.

Definition 11. Let $\varphi \in \mathcal{L}_M$ and $\Sigma \subseteq \mathcal{L}_M$. We say that φ is syntactically derived, or inferred, from Σ by using Armstrong-style (Simplification or U-Simplification) Axiomatic System if there is a sequence $(\varphi_i \mid 1 \leq i \leq n)$ such that $\varphi_n = \varphi$ and, for all $1 \leq i \leq n$, either $\varphi_i \in \Sigma$ or φ_i is obtained by applying one of the rules of Armstrong-style (Simplification or U-Simplification) Axiomatic System to implications belonging to $\{\varphi_j \mid 1 \leq j < i\}$. In this situation, the above sequence is said to be proof of the derivation.

From now on, $\Sigma \vdash_A \varphi$ denotes that φ is syntactically derived from Armstrong-style Axioms. In the same way, $\Sigma \vdash_S \varphi$ and $\Sigma \vdash_{SU} \varphi$ denote, respectively, that φ is syntactically derived from Σ by using Simplification or U-Simplification Axiomatic System.

The following lemma establishes that the three axiomatic systems are equivalent.

Lemma 12. *Let M be a set of attributes, $\Sigma \subseteq \mathcal{L}_M$ and $A \rightsquigarrow B \in \mathcal{L}_M$. We have that*

$$\Sigma \vdash_A A \rightsquigarrow B \quad \text{if and only if} \quad \Sigma \vdash_{SU} A \rightsquigarrow B \quad \text{if and only if} \quad \Sigma \vdash_S A \rightsquigarrow B.$$

To prove this lemma we show how each rule in one axiomatic system can be derived from the other two axiomatic systems.

Proof. First, we prove $\Sigma \vdash_A A \rightsquigarrow B$ implies $\Sigma \vdash_{SU} A \rightsquigarrow B$, and to do this, it is enough to prove that $[Un]$ and $[Simp]$ are derived rules from the Armstrong-style Axiomatic System.

The following sequence proves that $[Un]$ is obtained from Armstrong-style Axiomatic System:

$$\begin{aligned} \varphi_1 &= A \rightsquigarrow B && \text{By hypothesis.} \\ \varphi_2 &= A \rightsquigarrow C && \text{By hypothesis.} \\ \varphi_3 &= AC \rightsquigarrow B \sqcup C && \text{By [Augm] to } \varphi_1 \text{ with } C. \\ \varphi_4 &= A \rightsquigarrow A \sqcup C && \text{By [Augm] to } \varphi_2 \text{ with } A. \\ \varphi_5 &= A \rightsquigarrow B \sqcup C && \text{By [Trans] to } \varphi_4 \text{ and } \varphi_3. \end{aligned}$$

For [Simp], we provide the following proof:

$$\begin{array}{ll}
 \varphi_1 = A \rightsquigarrow B & \text{By hypothesis.} \\
 \varphi_2 = C \rightsquigarrow D & \text{By hypothesis.} \\
 \varphi_3 = A \sqcup (C \setminus B) \rightsquigarrow B \sqcup C & \text{Applying [Augm] to } \varphi_1 \text{ with } C \setminus B. \\
 \varphi_4 = B \sqcup C \rightsquigarrow B \sqcup D & \text{Applying [Augm] to } \varphi_2 \text{ with } B. \\
 \varphi_5 = A \sqcup (C \setminus B) \rightsquigarrow B \sqcup D & \text{Applying [Trans] to } \varphi_3 \text{ and } \varphi_4. \\
 \varphi_6 = B \sqcup D \rightsquigarrow D & \text{By [Inc].} \\
 \varphi_7 = A \sqcup (C \setminus B) \rightsquigarrow D & \text{Applying [Trans] to } \varphi_5 \text{ and } \varphi_6.
 \end{array}$$

Notice that the derivation of φ_3 is based on the weak dual Heyting structure, particularly the [wH4] property.

Second, we prove $\Sigma \vdash_{\mathcal{S}_U} A \rightsquigarrow B$ implies $\Sigma \vdash_{\mathcal{S}} A \rightsquigarrow B$, and to do this, it is enough to prove that [Key] is derived from \cup -Simplification Axiomatic System, and it is obtained with the following sequence:

$$\begin{array}{ll}
 \varphi_1 = A \rightsquigarrow b & \text{By hypothesis.} \\
 \varphi_2 = A \sqcup \bar{b} \rightsquigarrow A & \text{By [Inc].} \\
 \varphi_3 = A \sqcup \bar{b} \rightsquigarrow b & \text{Applying [Simp] to } \varphi_2 \text{ and } \varphi_1. \\
 \varphi_4 = A \sqcup \bar{b} \rightsquigarrow \bar{b} & \text{By [Inc].} \\
 \varphi_5 = A \sqcup \bar{b} \rightsquigarrow i & \text{Applying [Un] to } \varphi_3 \text{ and } \varphi_4.
 \end{array}$$

Notice that, once again, the algebraic structure is needed to derivate φ_3 . In this case, [wH3] is used.

Finally, to end the chain of equivalences, we have to prove that $\Sigma \vdash_{\mathcal{S}} A \rightsquigarrow B$ implies $\Sigma \vdash_A A \rightsquigarrow B$. Specifically, to do this it is enough to prove that [Trans] and [Augm] are derivated rules from Simplification Axiomatic System.

[Trans] is straightforwardly obtained from Simplification Axiomatic System because it is a particular case of [Simp] applied to $A \rightsquigarrow B$ and $B \rightsquigarrow C$ by using the Property [wH3].

To prove that [Augm] is obtained from Simplification Axiomatic System, we distinguish two cases depending on whether $B \sqcup C$ is i or not. On the one hand, if $B \sqcup C = i$, we have that there is a singleton x with $x \sqsubseteq B$ such that $\bar{x} \sqsubseteq C$. Then, the following sequence proves $A \rightsquigarrow B \vdash_{\mathcal{S}} A \sqcup C \rightsquigarrow i$:

$$\begin{array}{ll}
 \varphi_1 = A \rightsquigarrow B & \text{By hypothesis.} \\
 \varphi_2 = B \rightsquigarrow x & \text{By [Inc].} \\
 \varphi_3 = A \rightsquigarrow x & \text{Applying [Simp] to } \varphi_1 \text{ and } \varphi_2. \\
 \varphi_4 = A \sqcup \bar{x} \rightsquigarrow i & \text{Applying [Key] to } \varphi_3. \\
 \varphi_5 = A \sqcup C \rightsquigarrow A \sqcup \bar{x} & \text{By [Inc].} \\
 \varphi_6 = A \sqcup C \rightsquigarrow i & \text{Applying [Simp] to } \varphi_5 \text{ and } \varphi_4 \text{ and using [wH3].}
 \end{array}$$

On the other hand, the following sequence proves $A \rightsquigarrow B \vdash_{\mathcal{S}} A \sqcup C \rightsquigarrow B \sqcup C$ when $B \sqcup C \neq i$:

$$\begin{array}{ll}
 \varphi_1 = A \rightsquigarrow B & \text{By hypothesis.} \\
 \varphi_2 = B \sqcup C \rightsquigarrow B \sqcup C & \text{By [Inc].} \\
 \varphi_3 = A \sqcup ((B \sqcup C) \setminus B) \rightsquigarrow B \sqcup C & \text{Applying [Simp] to } \varphi_1 \text{ and } \varphi_2. \\
 \varphi_4 = C \rightsquigarrow \varepsilon & \text{By [Inc].} \\
 \varphi_5 = A \sqcup C \rightsquigarrow B \sqcup C & \text{Applying [Simp] to } \varphi_4 \text{ and } \varphi_3.
 \end{array}$$

In the last step, Proposition 4 and [wH1] have been used. \square

As a consequence of the above lemma and the soundness and completeness of Armstrong-style Axiomatic System, which was proved in [27], we have the following theorem.

Theorem 13. *The Simplification and \cup -Simplification Axiomatic Systems are sound and complete.*

Another direct consequence of Lemma 12 is that we can write $\Sigma \vdash \varphi$ without indicating the axiomatic system as a subscript.

5. The Simplification paradigm

The family of logics named Simplification has the common property that the inference rules can be seen as equivalence rules that allow simplifying a set of implications while preserving the knowledge, i.e., the set of implications that can be derived is the same.

First, we introduce the *generalized augmentation* rule, denoted by [gAug], that will be used in the proof of the equivalence rules: for all $A, B, C, D \in \mathfrak{3}^M$,

[gAug] If $A \sqsubseteq C$ and $D \sqsubseteq C \sqcup B$, then $A \rightsquigarrow B \vdash C \rightsquigarrow D$.

As we did before, we now check that [gAug] is derived from Simplification Axiomatic System.

Proposition 14. [gAug] is a derived inference rule from Simplification Axiomatic System.

Proof. The following sequence is a proof for [gAug]:

$\varphi_1 = A \rightsquigarrow B$ By hypothesis.
 $\varphi_2 = C \sqcup B \rightsquigarrow D$ By [Inc].
 $\varphi_3 = A \sqcup ((C \sqcup B) \setminus B) \rightsquigarrow D$ By using [Simp] to φ_1 and φ_2 .
 $\varphi_4 = C \rightsquigarrow A \sqcup ((C \sqcup B) \setminus B)$ By [Inc] and [wH1].
 $\varphi_5 = C \rightsquigarrow D$ By using [Simp] to φ_4 and φ_3 . \square

The following proposition gives the set of equivalences that allows simplifying a set of implications, i.e., to reduce the size of the set of implications, whereas the equivalence is preserved. By size of a set of implications Σ we mean

$$\|\Sigma\| = \sum_{A \rightsquigarrow B \in \Sigma} (|A| + |B|)$$

where $|i| = 1$ and $|X|$ is the sum of the cardinality of Pos(X) and Neg(X) for all $X \in \mathfrak{3}^U$.

Proposition 15. The following equivalence rules hold: for all $A, B, C, D \in \mathfrak{3}^M$,

[FragEq]: $\{A \rightsquigarrow B\} \equiv \{A \rightsquigarrow B \setminus A\}$.
 [UnEq]: $\{A \rightsquigarrow B, A \rightsquigarrow C\} \equiv \{A \rightsquigarrow B \sqcup C\}$.
 [ε -Eq]: $\{A \rightsquigarrow \varepsilon\} \equiv \emptyset$.
 [i -Eq]: $\{A \rightsquigarrow B\} \equiv \{A \rightsquigarrow i\}$ when $A \sqcup B = i$.
 [SimpEq]: $\{A \rightsquigarrow B, C \rightsquigarrow D\} \equiv \{A \rightsquigarrow B, C \setminus B \rightsquigarrow D \setminus B\}$ when $A \sqsubseteq C \setminus B$.

Proof. First, a proof for $A \rightsquigarrow B \vdash A \rightsquigarrow B \setminus A$ is the following sequence:

$\varphi_1 = A \rightsquigarrow B$ By hypothesis.
 $\varphi_2 = B \rightsquigarrow B \setminus A$ By [wH2] and [Inc].
 $\varphi_3 = A \rightsquigarrow B \setminus A$ Applying [Simp] to φ_1 and φ_2 using [wH3].

The opposite direction can be proved to apply [Augm] to $A \rightsquigarrow B \setminus A$ (which is the hypothesis) and using [wH4].

Second, to prove that from $\{A \rightsquigarrow B, A \rightsquigarrow C\}$ we can derive $\{A \rightsquigarrow B \sqcup C\}$ we use [Un] to both hypothesis. The opposite direction is straightforward from [gAug].

[ε -Eq] is due to [FragEq] and [Inc]. And [i -Eq] is due to [UnEq], [Inc] and [Frag].

Finally, the following sequence proves that from $\{A \rightsquigarrow B, C \rightsquigarrow D\}$ we can derive $\{A \rightsquigarrow B, C \setminus B \rightsquigarrow D \setminus B\}$ when $A \sqsubseteq C \setminus B$. We start proving that from $A \rightsquigarrow B$ and $C \rightsquigarrow D$ we derive $A \rightsquigarrow B$ and $C \setminus B \rightsquigarrow D \setminus B$ if $A \sqsubseteq C \setminus B$:

$\varphi_1 = A \rightsquigarrow B$ By hypothesis.
 $\varphi_2 = C \setminus B \rightsquigarrow C$ By using [gAug], $A \sqsubseteq C \setminus B$ and $C \sqsubseteq C \sqcup (C \setminus B)$.
 $\varphi_3 = C \rightsquigarrow D$ By hypothesis.
 $\varphi_4 = C \setminus B \rightsquigarrow D$ By using [Simp] to φ_2 and φ_3 .
 $\varphi_5 = C \setminus B \rightsquigarrow D \setminus B$ By using [gAug] to φ_4 and [wH2].

To prove the opposite direction, we use the following sequence.

$$\begin{aligned} \varphi_1 &= A \rightsquigarrow B && \text{By hypothesis.} \\ \varphi_2 &= C \setminus B \rightsquigarrow D \setminus B && \text{By hypothesis.} \\ \varphi_3 &= C \setminus B \rightsquigarrow D \sqcup B && \text{Applying [Un] to } \varphi_1 \text{ and } \varphi_2. \\ \varphi_4 &= C \rightsquigarrow D && \text{By using [gAug] to } \varphi_3 \text{ and [wH2].} \end{aligned}$$

Notice that in φ_3 we use $A \sqsubseteq C \setminus B$ and $(D \setminus B) \sqcup B = D \sqcup B$ by [wH4]. \square

In the following example, we apply these equivalences, left-to-right read, to reduce the size of the set of implications without losing any knowledge, that is, preserving the equivalences.

Example 2. Consider the set of weak implications Σ over the universe $U = \{a, b, c, d\}$, $\Sigma = \{b \rightsquigarrow c, i \rightsquigarrow b, bc \rightsquigarrow d\bar{a}, bd \rightsquigarrow c\bar{a}, bcd \rightsquigarrow i, \overline{abd} \rightsquigarrow c\bar{a}, c \rightsquigarrow d\}$. Let us see how the size of Σ can be reduced using the equivalences given in Proposition 15.

- By [FragEq] and [ε -Eq], we have that $\{i \rightsquigarrow b\} \equiv \{i \rightsquigarrow \varepsilon\} \equiv \emptyset$ and

$$\Sigma \equiv \{b \rightsquigarrow c, bc \rightsquigarrow d\bar{a}, bd \rightsquigarrow c\bar{a}, bcd \rightsquigarrow i, \overline{abd} \rightsquigarrow c\bar{a}, c \rightsquigarrow d\}$$

- Applying [SimpEq] and [UnEq], we have that

$$\{b \rightsquigarrow c, bc \rightsquigarrow d\bar{a}\} \equiv \{b \rightsquigarrow c, b \rightsquigarrow d\bar{a}\} \equiv \{b \rightsquigarrow cd\bar{a}\}$$

Therefore, $\Sigma \equiv \{b \rightsquigarrow cd\bar{a}, bd \rightsquigarrow c\bar{a}, bcd \rightsquigarrow i, \overline{abd} \rightsquigarrow c\bar{a}, c \rightsquigarrow d\}$.

- Applying [SimpEq] and [FragEq], we have that

$$\{b \rightsquigarrow cd\bar{a}, bd \rightsquigarrow c\bar{a}\} \equiv \{b \rightsquigarrow cd\bar{a}, b \rightsquigarrow \varepsilon\} \equiv \{b \rightsquigarrow cd\bar{a}\}$$

Therefore, $\Sigma \equiv \{b \rightsquigarrow cd\bar{a}, bcd \rightsquigarrow i, \overline{abd} \rightsquigarrow c\bar{a}, c \rightsquigarrow d\}$.

- Applying [SimpEq] and [UnEq], we have that

$$\{b \rightsquigarrow cd\bar{a}, bcd \rightsquigarrow i\} \equiv \{b \rightsquigarrow cd\bar{a}, b \rightsquigarrow i\} \equiv \{b \rightsquigarrow i\}$$

Then, $\Sigma \equiv \{b \rightsquigarrow i, \overline{abd} \rightsquigarrow c\bar{a}, c \rightsquigarrow d\}$.

- Finally, by [i -Eq], we obtain $\Sigma \equiv \{b \rightsquigarrow i, \overline{abd} \rightsquigarrow i, c \rightsquigarrow d\}$.

6. The automated reasoning method

In this section, we present how the introduced logic leads to the design of an automated reasoning method. To achieve this, we will first generalize the notion of syntactic closure of a set of attributes with respect to a set of implications. In the classical case, it allows us to check whether an implication $A \rightarrow B$ is derived or not from a set of implications Σ . To accomplish it, we check if the set B is contained in the syntactic closure of the A set with respect to Σ .

Definition 16. Let M be a finite set and $\Sigma \in \mathcal{L}_M$. The syntactic closure with respect to Σ is the map $[-]_{\Sigma} : \mathfrak{3}^M \rightarrow \mathfrak{3}^M$ defined as

$$[A]_{\Sigma} = \bigsqcup \{X \in \mathfrak{3}^M \mid \Sigma \vdash A \rightsquigarrow X\}.$$

The following theorem shows that the previous definition generalizes the so-called syntactic closure operator.

Theorem 17. Let M be a finite set and $\Sigma \in \mathcal{L}_M$. For any $A \rightsquigarrow B \in \mathcal{L}_M$ we have that $\Sigma \vdash A \rightsquigarrow B$ if and only if $B \sqsubseteq [A]_{\Sigma}$. In addition, the mapping $[-]_{\Sigma}$ is a closure operator in $\mathfrak{3}^M$.

Proof. It is straightforward that $\Sigma \vdash A \rightsquigarrow B$ implies $B \sqsubseteq [A]_{\Sigma}$, from the definition of $[A]_{\Sigma}$.

Assume that $B \sqsubseteq [A]_{\Sigma}$. On the one hand, by [Inc] we have that $\Sigma \vdash [A]_{\Sigma} \rightsquigarrow B$. On the other hand, as M is finite, the set $\chi = \{X \in \mathfrak{3}^M \mid \Sigma \vdash A \rightsquigarrow X\}$ is also finite and, from [Un], we have that $\Sigma \vdash A \rightsquigarrow \bigsqcup \chi$. Thus, from the Definition 16, we have that $\Sigma \vdash A \rightsquigarrow [A]_{\Sigma}$. Finally, applying [Simp] to $\Sigma \vdash A \rightsquigarrow [A]_{\Sigma}$ and $\Sigma \vdash [A]_{\Sigma} \rightsquigarrow B$, we have that $\Sigma \vdash A \rightsquigarrow B$.

Let us prove that $[-]_{\Sigma}$ is a closure operator in $\mathfrak{3}^M$. First, by [Inc], it is inflationary, i.e. $A \sqsubseteq [A]_{\Sigma}$ for all $A \in \mathfrak{3}^M$.

If $A \sqsubseteq B$ then, by [Inc], we have that $\Sigma \vdash B \rightsquigarrow A$ and, since $\Sigma \vdash A \rightsquigarrow [A]_{\Sigma}$ and using [Simp], we have that $\Sigma \vdash B \rightsquigarrow [A]_{\Sigma}$. Therefore, $[A]_{\Sigma} \sqsubseteq [B]_{\Sigma}$.

Finally, let us prove the idempotency of the $[-]_{\Sigma}$ mapping: $[[A]_{\Sigma}]_{\Sigma} \sqsubseteq [A]_{\Sigma}$ because $\Sigma \vdash A \rightsquigarrow [A]_{\Sigma}$, $\Sigma \vdash [A]_{\Sigma} \rightsquigarrow [[A]_{\Sigma}]_{\Sigma}$ and, by [Simp], $\Sigma \vdash A \rightsquigarrow [[A]_{\Sigma}]_{\Sigma}$. Therefore, since $[-]_{\Sigma}$ is inflationary, we have that $[-]_{\Sigma}$ is idempotent. \square

The second step necessary to achieve our goal of designing an automatic reasoning method based on Simplification logic is to obtain an analogous result to the deduction theorem for classical propositional logic, which established that $\Sigma \vdash \varphi \Rightarrow \psi$ if and only if $\Sigma \cup \{\varphi\} \vdash \psi$. Using the fact that any propositional formula χ is equivalent to $\top \Rightarrow \chi$ where \top denotes a tautology, the classical deduction theorem can be equivalently restated as $\Sigma \vdash \varphi \Rightarrow \psi$ if and only if $\Sigma \cup \{\top \Rightarrow \varphi\} \vdash \top \Rightarrow \psi$.

The automatic reasoning method we propose here is intended to answer the question of whether $a \rightsquigarrow b$ can be inferred from a theory Σ based on two pillars: one is a theorem of deduction reminiscent of propositional logic, and the other is a set of transformations that *simplify* the theory $\Sigma \cup \{\varepsilon \rightsquigarrow a\}$ by using Proposition 15, where the element $\varepsilon \in \mathfrak{3}^M$ will play the same role as the tautology \top of propositional logic.

Before giving the above-mentioned version of the deduction theorem, we introduce a notation and a previous result. For all $A \in \mathfrak{3}^M$:

- If $\varphi = X \rightsquigarrow Y$, then φ_A denotes $A \sqcup X \rightsquigarrow Y$.
- If $\Sigma \in \mathcal{L}_M$, then Σ_A denotes $\{\varphi_A : \varphi \in \Sigma\}$.

Lemma 18. *Let M be a finite set and $\Sigma \in \mathcal{L}_M$. For all $A \in \mathfrak{3}^M$ and $\varphi \in \mathcal{L}_M$,*

$$\Sigma \vdash \varphi \quad \text{implies} \quad \Sigma_A \vdash \varphi_A.$$

Proof. By Lemma 12, we can prove it by using Armstrong-style, Simplification or \cup -Simplification Axiomatic System. We consider here the last one. From Definition 11, $\Sigma \vdash \varphi$ if there is a sequence $\varphi_1, \dots, \varphi_n \in \mathcal{L}_M$ such that $\varphi_n = \varphi$ and, for all $1 \leq i \leq n$, either $\varphi_i \in \Sigma$ or φ_i is obtained by applying one of the rules of \cup -Simplification Axiomatic System to implications belonging to $\{\varphi_j \mid 1 \leq j < i\}$.

We prove by induction that, for all $1 \leq i \leq n$, we have that $\Sigma_A \vdash \varphi_{iA}$. If $\varphi_i \in \Sigma$, it is straightforward that $\varphi_{iA} \in \Sigma_A$. Assume, as induction Hypothesis, that $\Sigma_A \vdash \varphi_{jA}$ for all $1 \leq j < i$, and let $\varphi_i = B \rightsquigarrow C$ with $B, C \in \mathfrak{3}^M$, which is obtained by using one of the rules of \cup -Simplification Axiomatic System. We distinguish three cases:

- If φ_i is obtained by [Inc], then $C \sqsubseteq B$. Therefore, $C \sqsubseteq A \sqcup B$ and $\Sigma_A \vdash A \sqcup B \rightsquigarrow C$ also by [Inc].
- If φ_i is obtained by [Simp], then there exist $X \rightsquigarrow Y, Z \rightsquigarrow C \in \{\varphi_j \mid 1 \leq j < n\}$ such that $B = X \sqcup (Z \setminus Y)$. By the induction hypothesis, we have that $\Sigma_A \vdash A \sqcup X \rightsquigarrow Y$ and $\Sigma_A \vdash A \sqcup Z \rightsquigarrow C$.
 - If $A \sqcup B = i$, we have that $\Sigma_A \vdash A \sqcup B \rightsquigarrow C$ from [Inc].
 - If $A \sqcup B \neq i$ and $A \sqcup Z = i$, then $\Sigma_A \vdash i \rightsquigarrow C$, from item 1 in Proposition 8, since $A \sqcup B = A \sqcup X \sqcup (Z \setminus Y)$, we have that $A \sqcup X \sqcup Y = i$. Now, applying [i-Eq] to the hypothesis $\Sigma_A \vdash A \sqcup X \rightsquigarrow Y$, we have that $\Sigma_A \vdash A \sqcup X \rightsquigarrow i$. Thus, by [Simp] from $\Sigma_A \vdash A \sqcup X \rightsquigarrow i$ and $\Sigma_A \vdash i \rightsquigarrow C$, we obtain that $\Sigma_A \vdash A \sqcup X \rightsquigarrow C$. Finally, by using [gAug] we have that $\Sigma_A \vdash A \sqcup X \sqcup (Z \setminus Y) \rightsquigarrow C$, concluding $\Sigma_A \vdash A \sqcup B \rightsquigarrow C$.
 - If $A \sqcup B \neq i$ and $A \sqcup Z \neq i$, by item 2 in Proposition 8,

$$A \sqcup X \sqcup ((A \sqcup Z) \setminus Y) = A \sqcup X \sqcup (Z \setminus Y) = A \sqcup B$$

Now, by [Simp], from $\Sigma_A \vdash A \sqcup X \rightsquigarrow Y$ and $\Sigma_A \vdash A \sqcup Z \rightsquigarrow C$, we obtain that $\Sigma_A \vdash A \sqcup X \sqcup ((A \sqcup Z) \setminus Y) \rightsquigarrow C$, concluding also that $\Sigma_A \vdash A \sqcup B \rightsquigarrow C$.

- If φ_i is obtained by [\cup], then there are $B \rightsquigarrow X, B \rightsquigarrow Y \in \{\varphi_j \mid 1 \leq j < i\}$ such that $C = X \sqcup Y$. By induction hypothesis, $\Sigma_A \vdash A \sqcup B \rightsquigarrow X$ and $\Sigma_A \vdash A \sqcup B \rightsquigarrow Y$ applying [\cup] we obtain that $\Sigma_A \vdash A \sqcup B \rightsquigarrow X \sqcup Y$, concluding $\Sigma_A \vdash A \sqcup B \rightsquigarrow C$. \square

Now, we introduce and prove the deduction theorem.

Theorem 19. *Let M be a finite set and $\Sigma \in \mathcal{L}_M$. For any $A \rightsquigarrow B \in \mathcal{L}_M$:*

$$\Sigma \vdash A \rightsquigarrow B \quad \text{if and only if} \quad \Sigma \cup \{\varepsilon \rightsquigarrow A\} \vdash \varepsilon \rightsquigarrow B$$

Proof. Assume that $\Sigma \vdash A \rightsquigarrow B$. Then $\Sigma \cup \{\varepsilon \rightsquigarrow A\} \vdash \{\varepsilon \rightsquigarrow A, A \rightsquigarrow B\}$ and by using [Simp], we have $\Sigma \cup \{\varepsilon \rightsquigarrow A\} \vdash \varepsilon \rightsquigarrow B$.

For the converse implication, let us suppose that $\Sigma \cup \{\varepsilon \rightsquigarrow A\} \vdash \varepsilon \rightsquigarrow B$ then by Lemma 18 we have that $\Sigma_A \cup \{A \rightsquigarrow A\} \vdash A \rightsquigarrow B$. By [Inc], $\Sigma_A \cup \{A \rightsquigarrow A\} \equiv \Sigma_A$, so we have that $\Sigma_A \vdash A \rightsquigarrow B$. Now, observe that for all $\psi \in \Sigma_A$ there exists $\varphi \in \Sigma$ such that $\psi = \varphi_A$ and, by [gAug], we have that $\Sigma \vdash \varphi_A$. Therefore, $\Sigma \vdash A \rightsquigarrow B$. \square

Finally, as we have already advanced, we propose Algorithm 1, which, based on the deduction theorem (Theorem 19) and the equivalences of Proposition 15, computes the syntactic closure of a 3-set A with respect to a set of implications Σ . Furthermore, from Theorem 17, this algorithm also solves the deduction problem, i.e., it allows to discern whether $\Sigma \vdash A \rightsquigarrow B$ is satisfied or not.

Finally, let us prove that Algorithm 1 always ends, and it is sound and complete.

Theorem 20. *Let M be a finite set and $\Sigma \in \mathcal{L}_M$. For all $A \in \mathfrak{3}^M$, the algorithm returns $[A]_\Sigma$ and its cost is, in the worst case, $|\Sigma| \cdot \|\Sigma\|$.*

Algorithm 1: Syntactic closure of A with respect to Σ .

Input: Σ being a set of weak implications, A being a set of 3^U
Output: $[A]_\Sigma$
repeat
 $\Sigma_{old} := \Sigma; \Sigma := \emptyset$
 foreach $B \rightsquigarrow C \in \Sigma_{old}$ **do**
 $B_{new} := B \setminus A; C_{new} := C \setminus A$ // By [SimpEq]
 if $B_{new} = \varepsilon$ **then**
 $A := A \sqcup C_{new}$ // By [UnEq]
 else if $C_{new} \not\sqsubseteq B_{new}$ **then**
 Add $B_{new} \rightsquigarrow (C_{new} \setminus B_{new})$ to Σ // By [FragEq] and [ε -Eq]
 end
until $\Sigma = \Sigma_{old}$ or $A = i$
return A

Proof. Let A_0 and Σ_0 be the input parameters and A_j and Σ_j their values after the j -th iteration of the repeat loop. First, the algorithm ends because $\|\Sigma_{j-1}\| \geq \|\Sigma_j\|$ for each iteration j , and $\|\Sigma_{j-1}\| = \|\Sigma_j\|$ implies $\Sigma_{j-1} = \Sigma_j$, which is one of the stop conditions. Let n be the number of iterations that is lower or equal to $\|\Sigma\|$. In the worst case, the cost of the algorithm is $|\Sigma| \cdot \|\Sigma\|$.

At this point, we guarantee that we do not lose information in each step of the algorithm. Since only the equivalences given in Proposition 15 are used, we have that

$$\Sigma_{j-1} \cup \{\varepsilon \rightsquigarrow A_{j-1}\} \equiv \Sigma_j \cup \{\varepsilon \rightsquigarrow A_j\}$$

for all iteration j , and, by Theorems 19 and 17, we have that $\Sigma \vdash \varepsilon \rightsquigarrow A_n$ and $A_n \sqsubseteq [A]_\Sigma$. In order to prove the reverse inclusion, i.e. $[A]_\Sigma \sqsubseteq A_n$, we demonstrate that $\Sigma \vdash A \rightsquigarrow X$ implies $X \sqsubseteq A_n$.

Assume that $\Sigma \vdash A \rightsquigarrow X$, and $A_n \neq i$ because it is straightforward otherwise. By Theorem 19, we have that $\Sigma \vdash A \rightsquigarrow X$ is equivalent to $\Sigma \cup \{\varepsilon \rightsquigarrow A\} \vdash \varepsilon \rightsquigarrow X$ and, hence, to $\Sigma_n \cup \{\varepsilon \rightsquigarrow A_n\} \vdash \varepsilon \rightsquigarrow X$. Let $\varphi_1 \dots \varphi_k$ be a proof for $\Sigma_n \cup \{\varepsilon \rightsquigarrow A_n\} \vdash \varepsilon \rightsquigarrow X$. We prove by induction that, for all $1 \leq r \leq k$,

$$\text{if } \varphi_r = Y \rightsquigarrow Z \text{ and } Y \sqsubseteq A_n, \text{ then } Z \sqsubseteq A_n. \tag{4}$$

Notice that, by using the item 4 in Proposition 8, we have that:

$$A_n \wedge B = \varepsilon, \text{ for all } B \rightsquigarrow C \in \Sigma_n \tag{5}$$

In addition, by Algorithm 1, we have that:

$$B \neq \varepsilon \text{ and } C \neq \varepsilon, \text{ for all } B \rightsquigarrow C \in \Sigma_n \tag{6}$$

If $\varphi_r = Y \rightsquigarrow Z \in \Sigma_n \cup \{\varepsilon \rightsquigarrow A_n\}$ and $Y \sqsubseteq A_n$, then, by (5) and (6), $\varphi_r = \varepsilon \rightsquigarrow A_n$ and $Z = A_n$. Assume, as induction hypothesis, that φ_s satisfies (4), for all $1 \leq s < r$.

- If $\varphi_r = Y \rightsquigarrow Z$ is obtained by [Inc] and $Y \sqsubseteq A_n$, we have straightforwardly that $Z \sqsubseteq Y \sqsubseteq A_n$.
- If φ_r is obtained by [Simp] and $Y \sqsubseteq A_n$, then there exist $U \rightsquigarrow V, W \rightsquigarrow Z \in \{\varphi_s : 1 \leq s < r\}$ such that $Y = U \sqcup (W \setminus V) \sqsubseteq A_n$, which implies $U \sqsubseteq A_n$ and $W \setminus V \sqsubseteq A_n$. By induction hypothesis, we have that $V \sqsubseteq A_n$ and, by [wH4'], $W \sqsubseteq V \sqcup (W \setminus V) \sqsubseteq A_n$ concluding, by induction hypothesis, $Z \sqsubseteq A_n$.
- If $\varphi_r = Y \rightsquigarrow Z$ is obtained by [Un] and $Y \sqsubseteq A_n$, there exist $Y \rightsquigarrow V, Y \rightsquigarrow W \in \{\varphi_s : 1 \leq s < r\}$ such that $Z = V \sqcup W$. By induction hypothesis, $V \sqsubseteq A_n$ and $W \sqsubseteq A_n$ so we have that $Z = W \sqcup V \sqsubseteq A_n$.

Finally, since $\varphi_k = \varepsilon \rightsquigarrow X$ and $\varepsilon \sqsubseteq A_n$, we conclude that $X \sqsubseteq A_n$. \square

We conclude the section with an illustrative example of the Algorithm's execution.

Example 3. Let $M = \{a, b, c, d, e, f\}$ and let Σ be the following set of weak implications $\Sigma = \{\overline{a\bar{e}} \rightsquigarrow bc, cd \rightsquigarrow \overline{a\bar{b}}, i \rightsquigarrow \overline{e\bar{f}}, de \rightsquigarrow f, \overline{a} \rightsquigarrow de, \overline{b\bar{f}} \rightsquigarrow \overline{c\bar{a}b}, f \rightsquigarrow \overline{c}\}$. We show how Algorithm 1 computes $[adf]_\Sigma$.

First we have that $\Sigma_{old} = \Sigma, \Sigma = \emptyset$ and $A = adf$.

1. For $\overline{a\bar{e}} \rightsquigarrow bc \in \Sigma_{old}$, Algorithm 1 adds $\overline{e} \rightsquigarrow bc$ to Σ , having $\Sigma = \{\overline{e} \rightsquigarrow bc\}$.
 Notice that $\{\varepsilon \rightsquigarrow adf, \overline{a\bar{e}} \rightsquigarrow bc\} \equiv \{\varepsilon \rightsquigarrow adf, \overline{e} \rightsquigarrow bc\}$ by [SimpEq].
2. For $cd \rightsquigarrow \overline{a\bar{b}} \in \Sigma_{old}$, Algorithm 1 adds $c \rightsquigarrow \overline{a\bar{b}}$ to Σ . Thus, $\Sigma = \{\overline{e} \rightsquigarrow bc, c \rightsquigarrow \overline{a\bar{b}}\}$. Observe that $\{\varepsilon \rightsquigarrow adf, cd \rightsquigarrow \overline{a\bar{b}}\} \equiv \{\varepsilon \rightsquigarrow adf, c \rightsquigarrow \overline{a\bar{b}}\}$ by [SimpEq].
3. For $i \rightsquigarrow \overline{e\bar{f}} \in \Sigma_{old}$, Algorithm 1 does not change neither A nor Σ .
 Notice that, in this case, if we apply F, [FragEq] and [ε -Eq] we have $\{\varepsilon \rightsquigarrow adf, i \rightsquigarrow \overline{e\bar{f}}\} \equiv \{\varepsilon \rightsquigarrow adf\}$.
4. For $de \rightsquigarrow f$, Algorithm 1 changes neither A nor Σ .

In this case, applying also the equivalences [SimpEq], [FragEq] and [ϵ -Eq], we have $\{\epsilon \rightsquigarrow adf, de \rightsquigarrow f\} \equiv \{\epsilon \rightsquigarrow adf\}$.

5. For $\bar{a} \rightsquigarrow de \in \Sigma_{old}$, Algorithm 1 adds $\bar{a} \rightsquigarrow e$ to Σ , in such a case, we have that $\Sigma = \{\bar{e} \rightsquigarrow bc, c \rightsquigarrow \bar{a}\bar{b}, \bar{a} \rightsquigarrow e\}$.
By [SimpEq] we have $\{\epsilon \rightsquigarrow adf, \bar{a} \rightsquigarrow de\} \equiv \{\epsilon \rightsquigarrow adf, \bar{a} \rightsquigarrow e\}$.
6. For $\bar{b}f \rightsquigarrow \bar{c}ab \in \Sigma_{old}$, Algorithm 1 adds $\bar{b} \rightsquigarrow i$ to Σ , in such a case, we have $\Sigma = \{\bar{e} \rightsquigarrow bc, c \rightsquigarrow \bar{a}\bar{b}, \bar{a} \rightsquigarrow e, \bar{b} \rightsquigarrow i\}$. Observe that, if we apply [SimpEq] and [i -Eq] we have $\{\epsilon \rightsquigarrow adf, \bar{b}f \rightsquigarrow \bar{c}ab\} \equiv \{\epsilon \rightsquigarrow adf, \bar{b} \rightsquigarrow i\}$
7. For $f \rightsquigarrow \bar{c} \in \Sigma_{old}$, Algorithm 1 adds \bar{c} to A having $A = adf\bar{c}$. In this case, Σ does not change, having $\Sigma = \{\bar{e} \rightsquigarrow bc, c \rightsquigarrow \bar{a}\bar{b}, \bar{a} \rightsquigarrow e, \bar{b} \rightsquigarrow i\}$. Notice that, by applying [SimpEq] and [UnEq], we have $\{\epsilon \rightsquigarrow adf, f \rightsquigarrow \bar{c}\} \equiv \{\epsilon \rightsquigarrow adf\bar{c}\}$.
8. Now, the first iteration of the “repeat”-loop has finished and, as stated in the proof of Theorem 20, we have that

$$\begin{aligned} & \{\epsilon \rightsquigarrow adf\} \cup \Sigma_{old} = \\ & = \{\epsilon \rightsquigarrow adf, \bar{a}\bar{e} \rightsquigarrow bc, cd \rightsquigarrow \bar{a}\bar{b}, i \rightsquigarrow \bar{e}\bar{f}, de \rightsquigarrow f, \bar{a} \rightsquigarrow de, \bar{b}f \rightsquigarrow \bar{c}ab, f \rightsquigarrow \bar{c}\} \equiv \\ & \equiv \{\epsilon \rightsquigarrow adf\bar{c}, \bar{e} \rightsquigarrow bc, c \rightsquigarrow \bar{a}\bar{b}, \bar{a} \rightsquigarrow e, \bar{b} \rightsquigarrow i\} = \{\epsilon \rightsquigarrow adf\bar{c}\} \cup \Sigma \end{aligned}$$

9. Since $\Sigma_{old} \neq \Sigma$, Algorithm 1 changes Σ_{old} by Σ , sets Σ to \emptyset , and repeats the process, but does not modify neither A nor Σ (i.e. $\Sigma = \Sigma_{old}$). Therefore, Algorithm 1 finishes and returns $[A]_{\Sigma} = adf\bar{c}$.

7. Conclusions and future works

This paper aimed to introduce a new logic, based on the Simplification paradigm, for reasoning about the kind of implications presented [27]. Those implications allow capturing knowledge in contexts with missing information, the so-called partial formal contexts. Even though the underlying algebraic framework introduced in [27] is not based on a complete dual Heyting algebra, which was established as a minimum requirement by the extension of the Simplification Logic given in [40], we have introduced a new weaker algebraic structure where this has been possible. Specifically, we have introduced the so-called weak dual Heyting algebra and proved that it is enough to extend the Simplification Logic to this new framework ensuring its soundness and completeness.

As a common feature of the family of so-called Simplification logics, we prove that in this case too, inference rules can be described as equivalence rules. These equivalence rules allow reducing the size of implicational systems without loss of knowledge, i.e., simplifying implicational systems by removing redundant information.

This result, together with Theorems 17 and 19 that resemble the so-called classical deduction theorem, allowed us to provide an algorithm for computing the syntactic closure and, consequently, for defining an automated reasoning method about weak implications. This Algorithm follows the same schema as those proposed for other extensions of the Simplification Logic [40–42]. Finally, we have proved the correctness of the Algorithm and studied its cost in the worst case.

In future works, we will generalize these results by considering other information interpretations and looking for a unified framework for reasoning with missing information. In particular, we would like to combine our work with the lines opened in [4] to properly define automated methods for an FCA disjunctive logic and in [30] to move our logic to the fuzzy framework. Another direction we have already opened is to define another interpretation of the missing information, considering all possible information and introducing a new notion of an implication that captures such interpretation. Specifically, we will extend this paper by considering two fuzzy values to represent the positive and negative knowledge, following Atanassov’s approach to fuzzy sets.

CRedit authorship contribution statement

Francisco Pérez-Gómez: Term, Conceptualization, Methodology, Writing-Original draft preparation, Investigation. **Pablo Cordero:** Term, Conceptualization, Methodology, Writing-Original draft preparation, Investigation. **Manuel Enciso:** Term, Conceptualization, Methodology, Writing-Original draft preparation, Investigation. **Ángel Mora:** Term, Conceptualization, Methodology, Writing-Original draft preparation, Investigation.

Declaration of competing interest

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Data availability

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References

- [1] R. Wille, Restructuring lattice theory: an approach based on hierarchies of concepts, in: *Ordered Sets*, vol. 83, 1982, pp. 445–470.
- [2] B. Ganter, S. Obiedkov, *Conceptual Exploration*, Springer, Berlin, 2016.
- [3] B. Ganter, R. Wille, *Formal Concept Analysis: Mathematical Foundations*, Springer, Berlin, 1996.
- [4] D. Dubois, J. Medina, H. Prade, E. Ramírez-Poussa, Disjunctive attribute dependencies in formal concept analysis under the epistemic view of formal contexts, *Inf. Sci.* 561 (2021) 31–51.
- [5] S.O. Kuznetsov, Complexity of learning in concept lattices from positive and negative examples, *Discrete Appl. Math.* 142 (2004) 111–125.
- [6] P. Butka, J. Pócs, J. Pócsová, Representation of fuzzy concept lattices in the framework of classical FCA, *J. Appl. Math.* (2013).
- [7] S.O. Kuznetsov, Mathematical aspects of concept analysis, *J. Math. Sci.* 80 (1996) 1654–1698.
- [8] S.O. Kuznetsov, A. Revenko, Interactive error correction in implicative theories, *Int. J. Approx. Reason.* 63 (2015) 89–100.
- [9] B. Ganter, L. Kwuida, Which concept lattices are pseudocomplemented?, in: *Lecture Notes in Computer Science*, vol. 3403, 2005, pp. 408–416.
- [10] R. Missaoui, L. Nourine, Y. Renaud, Computing implications with negation from a formal context, *Fundam. Inform.* 115 (4) (2012) 357–375.
- [11] B. Ganter, S.O. Kuznetsov, Hypotheses and version spaces, in: *Lecture Notes in Computer Science*, vol. 2746, 2003, pp. 83–95.
- [12] E. Bartl, J. Konecny, L-concept lattices with positive and negative attributes: modeling uncertainty and reduction of size, *Inf. Sci.* 472 (2019) 163–179.
- [13] J.M. Rodríguez-Jiménez, P. Cordero, M. Enciso, S. Rudolph, Concept lattices with negative information: a characterization theorem, *Inf. Sci.* 369 (2016) 51–62.
- [14] J.M. Rodríguez-Jiménez, P. Cordero, M. Enciso, A. Mora, Data mining algorithms to compute mixed concepts with negative attributes: an application to breast cancer data analysis, *Math. Methods Appl. Sci.* 39 (16) (2016) 4829–4845.
- [15] P. Cordero, M. Enciso, A. Mora-Bonilla, J. Rodríguez-Jimenez, Inference of mixed information in formal concept analysis, *Stud. Comput. Intell.* 796 (2019) 81–87.
- [16] R. Holzer, Knowledge acquisition under incomplete knowledge using methods from formal concept analysis: part II, *Fundam. Inform.* 63 (1) (2004) 41–63.
- [17] P. Burmeister, R. Holzer, Treating incomplete knowledge in formal concept analysis, in: *Lecture Notes in Computer Science*, vol. 3626, 2005, pp. 114–126.
- [18] S.A. Obiedkov, Modal logic for evaluating formulas in incomplete contexts, in: *Lecture Notes in Computer Science*, vol. 2393, 2002, pp. 314–325.
- [19] Z. Huilai, L. Jinhai, Granule description based knowledge discovery from incomplete formal contexts via necessary attribute analysis, *Inf. Sci.* 485 (2019) 347–361.
- [20] B. Long, W. Xu, X. Zhang, Double threshold construction method for attribute-induced three-way concept lattice in incomplete fuzzy formal context, *J. Eng.* 2020 (2020) 549–554.
- [21] B. Ganter, C. Meschke, A formal concept analysis approach to rough data tables, in: *Transactions on Rough Sets XIV*, Springer-Verlag, Berlin, Heidelberg, 2011, pp. 37–61.
- [22] Y. Yiyu, Interval sets and three-way concept analysis in incomplete contexts, *Int. J. Mach. Learn. Cybern.* 20 (8) (2017) 3–20.
- [23] H. Zhi, J. Qi, T. Qian, R. Ren, Conflict analysis under one-vote veto based on approximate three-way concept lattice, *Inf. Sci.* 516 (2020) 316–330.
- [24] B. Bogdanović, N. Veljković, M. Frtunić, D. Puflović, L. Stoimenov, On revealing shared conceptualization among open datasets, *J. Web Semant.* 66 (2021) 100624.
- [25] L.D.C.S. Subhashini, Y. Li, J. Zhang, A. Atukorale, Integration of fuzzy logic and a convolutional neural network in three-way decision-making, *Expert Syst. Appl.* 202 (2022) 117103.
- [26] D. Ciucci, D. Dubois, J. Lawry, Borderline vs. unknown: comparing three-valued representations of imperfect information, *Int. J. Approx. Reason.* 55 (9) (2014) 1866–1889.
- [27] F. Pérez-Gómez, P. Cordero, M. Enciso, A. Mora, A new kind of implication to reason with unknown information, in: *Lecture Notes in Computer Sciences*, vol. 12733, 2021, pp. 74–90.
- [28] Á. Mora, M. Enciso, P. Cordero, I. Pérez de Guzmán, An efficient preprocessing transformation for functional dependencies sets, in: *Lecture Notes in Artificial Intelligence*, vol. 3040, 2004, pp. 136–146.
- [29] A. Mora, P. Cordero, M. Enciso, I. Fortes, G. Aguilera, Closure via functional dependence simplification, *Int. J. Comput. Math.* 89 (4) (2012) 510–526.
- [30] J. Konecny, Attribute implications in L-concept analysis with positive and negative attributes: validity and properties of models, *Int. J. Approx. Reason.* 120 (2020) 203–215.
- [31] A. Buzmakov, A. Napoli, How fuzzy FCA and pattern structures are connected?, in: 5th Workshop “What can FCA do for Artificial Intelligence?”, 2016.
- [32] T. Quan, S.C. Hui, T.H. Cao, A fuzzy FCA-based approach to conceptual clustering for automatic generation of concept hierarchy on uncertainty data, in: *CLA 2004*, 2004, pp. 1–12.
- [33] O. Kridlo, M. Ojeda-Aciego, Extending formal concept analysis using intuitionistic L-fuzzy sets, in: 2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 2017, pp. 1–6.
- [34] J. Slupecki, G. Bryll, T. Prucnal, Some remarks on three-valued logic of J. Lukasiewicz, *Stud. Log.* 21 (1) (1967) 45–66.
- [35] S.C. Kleene, *Introduction to Metamathematics*, Elsevier, 1980.
- [36] E.F. Codd, Extending the database relational model to capture more meaning, *ACM Trans. Database Syst.* 4 (4) (1979) 397–434.
- [37] N.D. Belnap, *A Useful Four-Valued Logic*, Springer, Netherlands, Dordrecht, 1977, pp. 5–37.
- [38] D. Dubois, S. Konieczny, H. Prade, Quasi-possibilistic logic and its measures of information and conflict, *Fundam. Inform.* 57 (2–4) (2003) 101–125.
- [39] W.W. Armstrong, Dependency structures of data base relationships, in: *IFIP Congress*, 1974, pp. 580–583.
- [40] R. Belohlavek, P. Cordero, M. Enciso, A. Mora, V. Vychodil, Automated prover for attribute dependencies in data with grades, *Int. J. Approx. Reason.* 70 (2016) 51–67.
- [41] E. Rodríguez-Lorenzo, P. Cordero, M. Enciso, R. Missaoui, A.M. Bonilla, An axiomatic system for conditional attribute implications in triadic concept analysis, *Int. J. Intell. Syst.* 32 (8) (2017) 760–777.
- [42] P. Cordero, M. Enciso, A. Mora, V. Vychodil, Parameterized simplification logic I: reasoning with implications and classes of closure operators, *Int. J. Gen. Syst.* 49 (7) (2020) 724–746.